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CLOGGING OF MULTIGRAPHS AS TOY MODELS OF FILTERS*

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Abstract. In this paper we model filters as networks of channels. As suspension (fluid with particles) flows through the filter, particles are trapped and clog channels. We assume there is no flow through clogged channels. The filter becomes impermeable after not all, but only a portion, of the channels clog. In this paper we compute an upper bound on the number of channels that clog. This bound is a function of properties of the network. Our results provide an understanding of the relationship between the filter pore space geometry and the filter efficiency.

Key words. filters, porous media, clogging, networks, graph

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1. Introduction.

A porous medium or porous material is a solid, often called matrix, permeated by an interconnected network of space filled with fluid. These fluid-filled spaces are called pores. A porous material is said to be permeable if fluid can flow through its pores from one side to an opposite side of the material. Filters are examples of porous materials.

Fluid suspensions, or suspensions for short, are fluids with small solid particles in them. As suspension flows through a permeable porous material, some particles are trapped within the material. In fact, the function of the filters we consider in this paper is to *clean* suspensions by capturing particles.

The removal of particles from fluid suspensions is of importance in a wide range of industrial and technological applications such as wastewater treatment [20], drinking water treatment, and other filtration processes [4, 35]. Our studies are motivated by the filters used in the process known as deep bed filtration. In this process, as suspension flows through a filter, particles penetrate the filter and deposit at various depths [36]. As a result, the fluid suspension is cleaner when it exits the filter (i.e., the suspension exits the filter with fewer solid particles than it originally had when it entered the filter).

Mathematical models for studying transport in porous media can be classified as either macroscale models [5, 15, 24, 25, 26, 28, 36] or pore-scale models (also known as microscale models) [9, 21, 31]. Macroscale models are those that model phenomena occurring at length scales much larger than the pore dimensions. On the other hand, pore-scale models, as the name suggests, model phenomena occurring at the length scale of the size of the pores. Within the pore-scale models, the class of network models is very popular. Network models are a class of models that represent the pore space in an idealized fashion, which can be channels, or pore bodies connected by pore throats. Our work belongs to the class of network models.

Network models for studying transport in porous media were introduced by Fatt [10, 11, 12]. Donaldson [8] was the first one to use networks to study particle transport within porous media. The clogging of particles has been studied

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in networks with different geometries including bundle of parallel tubes [8], square networks [14, 16, 23], triangular networks [3, 27], cubic networks [2, 17, 33], bubble models [6, 22], and the so-called three-dimensional physically representative networks [1, 34].

Consider a three-dimensional filter that is a finite network of channels. These channels need not be straight. Assume the filter is located between two parallel planes, a top and a bottom. We will refer to the top and bottom planes as the top and bottom boundaries of the filter, respectively. As suspension flows through the filter from its bottom boundary to its top boundary, particles clog channels. Assume suspension cannot flow through clogged channels. In this work, channels that are not clogged are said to be open. Note that there can only be flow through channels that are part of a path of open channels that connects the bottom and top boundaries of the filter. As channels clog, some paths of open channels connecting the two boundaries are broken. Thus, suspension stops flowing not only through the clogged channels but also through some open channels, i.e., those that are no longer part of a path of open channels connecting the two boundaries. The filter will become impermeable when no connected path of open channels exists between the bottom and top boundaries. This occurs after not all, but only a fraction, of the channels clog. In this paper we find an upper bound of this number. Our upper bound is a function of the geometry of the network. In particular, we are able to identify the filter geometries (in this idealized network context) for which the largest fraction of channels may be clogged before the filter ceases to be permeable. Our results suggest that filters with these geometries may have longer lives than others.

Most of the work on clogging in networks that can be found in the literature consists of simulations of the suspension dynamics within the medium. Our work is an analysis that is independent of the dynamics; that is, it depends only on the topology of the network. On the other hand, there are connections, but also many differences, between our work and the theory of bond percolation [13, 29]. In percolation theory, channels or edges are removed randomly and independently of each other. Here, channels clog, but neither randomly nor independently of each other. It is important to note the order in which they clog.

The problem considered in this paper was previously studied in [18, 19] for two-dimensional networks. In those works, the authors use graph theory techniques that make sense but are valid only for planar graphs. In particular, the results obtained did not have obvious three-dimensional extension. Here, we use a completely different strategy that turns out to be somewhat simpler and certainly more general. We are able to consider both two-dimensional and three-dimensional networks. When the networks are two-dimensional, the results in [18, 19] are recovered. Our results for three-dimensional networks are new.

This paper is organized as follows. In section 2, we describe the filters as networks. In section 3, we review the elements of graph theory that are needed in the rest of this paper. In section 4, we describe the geometry of our filters and the meaning of clogged edges. In section 5, we motivate our work and state our upper bound. In section 6, we prove the validity of our upper bound. In section 7, we show that our upper bound is sharp in a sense that is described in that section. In section 8, we obtain an alternative description of our bound. In section 9, we consider a three-dimensional example. In section 10, we obtain an alternative description of our bound valid for two-dimensional networks and we make the connection with the work in [18, 19]. A final discussion is given in section 11.

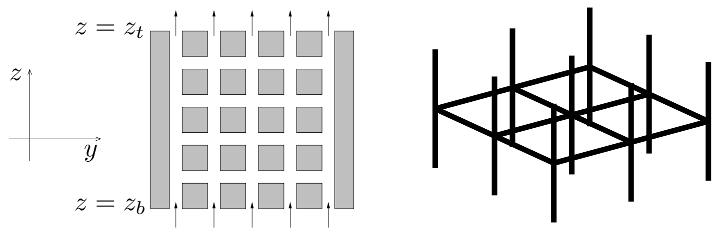

FIG. 1. *The left and right figures show a two- and three-dimensional network of channels, respectively. As illustrated in the figures, we use the standard Cartesian coordinate systems. The arrows in the networks indicate the direction of the flow.*

2. Filters as networks. As illustrated in Figure 1, we model filters as two- or three-dimensional networks of channels (not necessarily straight). These networks connect two parallel lines in the two-dimensional case or two parallel planes in the three-dimensional case. We will refer to these lines or planes as the top and bottom boundaries of the network. The pores are the interior of the channels. As illustrated in the figure, the bottom and top boundaries are located at $z = z_b$ and $z = z_t$, respectively.

In our model, channels are either open or clogged. Suspension can flow only through open channels. There is no flow through clogged channels. Within an open channel, suspension flows from the end with higher pressure to the opposite end. If both ends are at the same pressure, there is no flow within the channel.

We assume that suspension can flow into the filter only through the bottom boundary at $z = z_b$, and can flow out of the filter only through the top boundary at $z = z_t$. Both fluid and particles are assumed to be incompressible, and thus volume of suspension enters the filter through the bottom boundary at the same rate it exits the filter through the top boundary.

We assume that the bottom boundary is held at constant pressure $p = p_b$ and the top boundary at $p = p_t$, where $p_b > p_t$. Note that the filter is permeable if and only if there is a path of open channels connecting the bottom boundary with the top boundary. Due to the difference in pressure between the top and bottom boundaries, there is flow through the filter if and only if the filter is permeable.

We assume that initially all the channels are open. As suspension flows through the filter, particles are trapped, causing channels to clog, i.e., channels change from open to clogged. Eventually, the filter is no longer permeable. Note that an open channel can clog only if there is flow through it. We assume that different channels do not clog simultaneously.

ASSUMPTIONS 2.1. *For future reference, we list here the key assumptions of our model.*

1. *Channels are either open or clogged.*
2. *There is no flow through clogged channels.*
3. *Suspension can flow into the filter only through the bottom boundary and out of the filter only through the top boundary.*
4. *Fluid and particles are incompressible.*
5. *Initially all the channels are open.*
6. *An open channel may clog if there is flow through it.*
7. *An open channel does not clog if there is no flow through it.*
8. *Different channels do not clog simultaneously.*
9. *Once a channel clogs, it remains clogged.*

3. Review of concepts in graph theory. In this section we review concepts of graph theory that we need in the rest of the paper. More details on graph theory can be found in [7].

A *graph* G consists of a nonempty set of elements, called *nodes*, and a list of unordered pairs of these elements, called *edges*. We identify each node with a different point in the plane or space and each edge with a line (not necessarily straight) joining its two nodes without intersecting any other node. If e is an edge joining the two nodes a and b, we say that a and b are the end points of e and that e connects a and b. For convenience we take each edge e to be a closed set, i.e., e includes its end points. If $a = b$, i.e., the end points of an edge e are the same, we say that e is a loop. In a graph, two different edges do not have the same pair of end points. We have a *multigraph* when this restriction is removed, i.e., in a multigraph, two different edges can have the same end points.

We say that two nodes a and b are connected if there exists a sequence of nodes $n_0, n_1, \ldots, n_k$ such that $a = n_0$, $b = n_k$, and for each $1 \leq i \leq k$ there exists an edge e_i that connects n_{i-1} and n_i. In this case, the alternating sequence of nodes and edges $n_0, e_1, n_1, e_2, n_2, \ldots, e_k, n_k$ forms a *walk between a and b* or simply a *walk*. We say that $a = n_0$ and $b = n_k$ are the end points of the walk. If $n_i \neq n_j$ for all $i \neq j$, we say that the walk is a *path*. If $n_0 = n_k$ and $n_i \neq n_j$ for $i < j$ except when $(i, j) = (0, k)$, we say that the walk is a *cycle*. We will identify each walk with the curve in the plane or space formed by its edges.

Let G be a multigraph. S is a *submultigraph* of G if S is a multigraph and S is included in G; i.e., every node of S is also a node of G and every edge of S is also an edge of G.

A multigraph is *connected* if there is a walk between any pair of its nodes, and is *disconnected* otherwise. Every multigraph is the union of pairwise disjoint connected submultigraphs (i.e., no pair of these submultigraphs have a node in common). Each of these submultigraphs is called a *connected component* of the multigraph.

The degree of a node n, that we denote by d_n, is the number of edges that have n as an end point, where the loops are counted twice. The average degree of a multigraph G, that we denote by d_G, is defined as the average of the degrees of the nodes of G, $d_G = n_G^{-1} \sum d_n$, where the sum is over all nodes n and n_G is the number of nodes of G. Note that

$$d_G = 2\frac{e_G}{n_G}, \tag{3.1}$$

where e_G is the number of edges of G. Examples of two- and three-dimensional multigraphs, which are actually graphs, are shown in Figure 2.

4. Geometry of the filters and clogged edges. Recall that, as described in section 2, filters are modeled as networks in this paper. Thus, when we mention filters, we are referring to the networks described in section 2.

To each filter we associate a multigraph in a natural way. The edges are the channels and the nodes the end points of the edges.

Recall that the bottom and top boundaries of the filter are located at $z = z_b$ and $z = z_t$, respectively. Thus, the multigraph is included in $z_b \leq z \leq z_t$. Note that there are nodes in the bottom and top boundaries.

We consider filters with a finite number of channels. Thus, our multigraphs are finite multigraphs, i.e., they contain a finite number of nodes and edges. As examples, in Figure 2 we show the multigraphs associated with the filters of Figure 1.

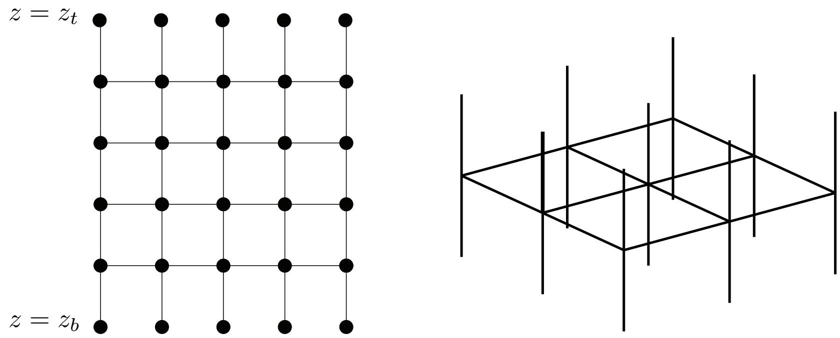

FIG. 2. *A two-dimensional multigraph (left figure) and a three-dimensional multigraph (right figure). The black small circles are the nodes and the solid lines the edges. The nodes are not shown in the three-dimensional multigraph.*

DEFINITION 4.1. *We say that a node is an exterior node if it is located at $z = z_b$ or $z = z_t$. Otherwise, we say that the node is an interior node.*

Suspension enters the network through exterior nodes at $z = z_b$ and exits the network through exterior nodes at $z = z_t$.

Naturally, we say that an edge is clogged (open) if the corresponding channel is clogged (open).

We study our filter at a fixed time. In other words, when we say that an edge is clogged, we mean that the edge is clogged at that fixed time. Analogously, when we say that an edge is open, we mean open at that fixed time.

5. Motivation and statement of the main result. In this section we will discuss questions that motivate our work and we will state our main result. We start with the following definition.

DEFINITION 5.1. *Let G be the multigraph of one of our filters. Let $e_1, e_2, \ldots, e_s$ be edges of G. We say that $e_1, e_2, \ldots, e_s$ is a sequence of edges that may clog if, for each $1 \leq i \leq s$, there is flow through the edge e_i when the edges $e_1, e_2, \ldots, e_{i-1}$ are clogged and all the other edges of G are open. We say that s is the length of the sequence.*

Note that, as its name suggests, if $e_1, e_2, \ldots, e_s$ is a *sequence of edges that may clog*, it is possible that these edges do clog and in that order, i.e., e_i is the ith edge to clog. The converse is also true if $e_1, e_2, \ldots, e_s$ are the clogged edges and are labeled in the order in which they clogged, i.e., e_i was the ith edge to clog; then, according to Definition 5.1, $e_1, e_2, \ldots, e_s$ is a *sequence of edges that may clog*.

Examples of sequences of edges that may clog are given in Figure 3. As that figure illustrates, any filter has many sequences of edges that may clog.

Assume now that, as suspension flows through a filter, particles clog some edges and the filter becomes nonpermeable after t edges clog. Let $e_1, e_2, \ldots, e_t$ be the clogged edges labeled in the order in which they clogged, i.e., e_i was the ith edge to clog. Note that, according to Definition 5.1, $e_1, e_2, \ldots, e_t$ is a sequence of edges that may clog. Two possible examples of such sequences are shown in the left and middle figures of Figures 3.

We expect the number of particles that are trapped to increase with the number of clogged edges t, which is the length of the sequence $e_1, e_2, \ldots, e_t$. In the left figure of Figure 3, $t = 4$. On the other hand, $t = 13$ in the middle figure. The goal is to

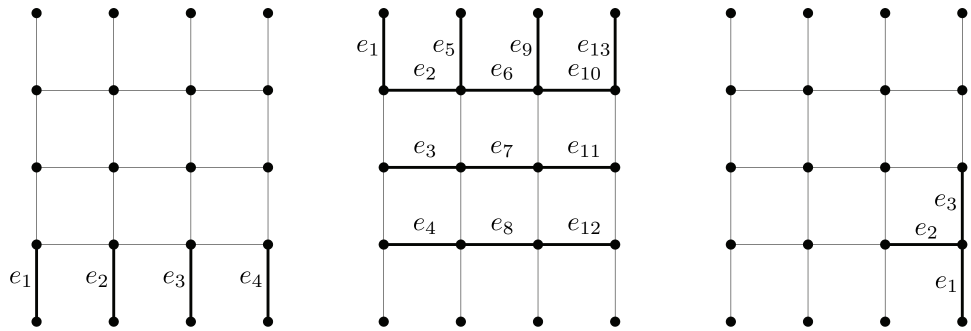

FIG. 3. *The sequences $e_1, \ldots, e_r$ in the left and middle figures ($r = 4$ in the left figure and $r = 13$ in the middle figure) are two sequences of edges that may clog. The sequence in the right figure is not a sequence that may clog. Once e_1 and e_2 clog, there is no more flow through e_3, and thus, e_3 cannot clog if e_1 and e_2 are clogged.*

trap as many particles as possible, and thus the situation in the middle figure is more desirable than that in the left figure. A natural question that arises is whether, for the filter of Figure 3, there is another sequence of edges that may clog of longer length. In other words, can t be larger than 13? More generally: for a given filter, what is the length of the longest sequence of edges that may clog?

Seeking to answer the above questions, in the next section we will prove the following theorem.

THEOREM 5.2. *Let G be the multigraph that corresponds to one of our filters. If every connected component of G contains an exterior node (see Definition* 4.1*), then*

$$\#\{\textit{clogged edges}\} \leq \#\{\textit{edges of } G\} - \#\{\textit{interior nodes of } G\}, \tag{5.1}$$

where # means "number of."

Note that there cannot be flow through the connected components of G that do not contain any exterior nodes. Thus, none of the edges that belong to connected components of G that do not contain any exterior nodes can clog. As a consequence, we can apply our theorem to any filter. We should first simply remove the connected components of the associated multigraph that do not contain any exterior node before applying the theorem.

In the example of Figure 3, we have that $\#\{\text{edges of } G\} = 25$, $\#\{\text{interior nodes of } G\} = 12$, and thus any sequence of edges that may clog has at most a length of 13. As a consequence, there is no sequence of edges that may clog longer than that of the middle figure.

6. Upper bound on the number of clogged channels. For future reference, we start by stating the following elementary observations that can be easily proved.

OBSERVATION 6.1. *Let G_1 and G_2 be two multigraphs. If G_1 is a submultigraph of G_2, then each connected component of G_1 is the submultigraph of a connected component of G_2.*

OBSERVATION 6.2. *Let G be a connected multigraph. Let e be an edge of G with end points a and b. Let G' be the multigraph that results from removing e from G. Let C_a and C_b be the connected components of G' that contain the nodes a and b, respectively. Then $G' = C_a \cup C_b$, and thus, G' has at most two connected components (it is possible that $G' = C_a = C_b$).*

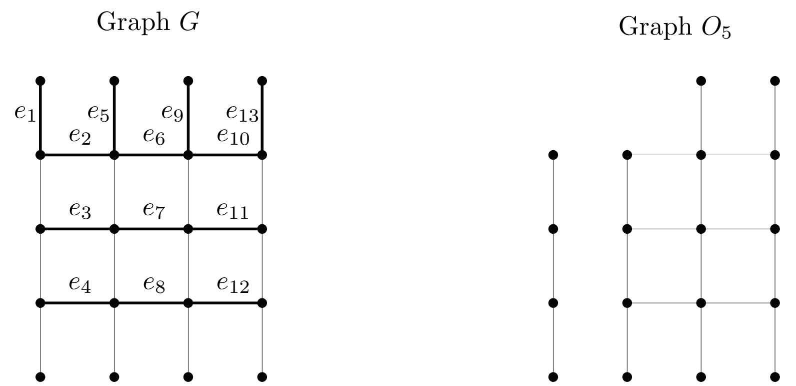

FIG. 4. *Example illustrating Definition* 6.2. *The clogged edges are* $e_1, \ldots, e_{13}$ *(labeled in the order in which they clogged). The left figure is the multigraph* G. *The right figure is the multigraph* O_5.

OBSERVATION 6.3. *Let* G *be a connected multigraph. Let* n_G *and* e_G *be the number of nodes and edges of* G*, respectively. Then,* $e_G \geq n_G - 1$.

DEFINITION 6.1. *Let* G *be the multigraph of one of our filters. We say that* P *is a percolating path if* P *is a path that connects the top and bottom boundaries and if the only nodes of* P *in the boundaries are its end points.*

OBSERVATION 6.4. *Let* G *be the multigraph of one of our filters and* e *an edge of* G. *If there is flow through* e*, then* e *belongs to a percolating path of open edges.*

Having stated the above observations, we now proceed with the necessary steps to bound the number of clogged edges. We start with the following definition.

DEFINITION 6.2. *Here, as in the rest of this section,* G *is a multigraph that corresponds to one of our filters. Let* t *be the number of clogged edges. Let* $e_1, e_2, \ldots, e_t$ *be the clogged edges labeled in the order in which they clogged, i.e.,* e_i *was the* i*th edge to clog. For each* k *that satisfies* $1 \leq k \leq t$*, we denote by* O_k *the multigraph of open edges by the time the* k*th edge clogged. More precisely,*

$$\{\textit{edges of } O_k\} = \{\textit{edges of } G\} - \{e_1, \ldots, e_k\}, \tag{6.1}$$

and the nodes of O_k *are the end points of the edges of* O_k. *In other words,* O_k *is obtained by removing* e_k *from* O_{k-1} *and any node left isolated.*

Note that for each k, O_k is a submultigraph of G. For convenience, we define $O_0 = G$. An example illustrating the definition of O_k is given in Figure 4.

OBSERVATION 6.5. *For each* k $(0 < k \leq t)$*,* e_k *belongs to a percolating path that is included in* O_{k-1}.

Proof. Since e_k clogs after $e_1, \ldots, e_{k-1}$ clogged, there is flow through e_k before it clogs and after $e_1, \ldots, e_{k-1}$ clogged. Thus, e_k belongs to a percolating path P that does not contain any of the edges $e_1, \ldots, e_{k-1}$, and thus P is a path in O_{k-1}. ∎

We now proceed with a key observation.

OBSERVATION 6.6. *Assume that every connected component of* G *contains an exterior node (see Definition* 4.1 *for the meaning of exterior node). Let* k *be an integer that satisfies* $0 \leq k \leq t$. *Then, each connected component of* O_k *contains an exterior node.*

Proof. Let C_k be a connected component of O_k. It follows from Observation 6.1 that there exists $C_0, C_1, C_2, \ldots, C_{k-1}$ such that, for each i, $0 \leq i \leq k$, C_i is a connected component of O_i and for each i, $1 \leq i \leq k$, $C_i \subseteq C_{i-1}$. Note that if e_i is an edge of C_{i-1}, then $C_i \neq C_{i-1}$. (This is because O_i is obtained by removing e_i from O_{i-1}.)

Since C_0 is a connected component of G, C_0 contains an exterior node. If $C_k = C_0$, then C_k contains an exterior node. Otherwise, let j be such that $1 \leq j \leq k$, $C_{j-1} \neq C_j$, and $C_j = C_k$. Thus, C_k is a connected component of the multigraph that results from removing e_j from C_{j-1}. Note that e_j clogged while being an edge of C_{j-1}. This implies that e_j belongs to a percolating path P that is included in C_{j-1}. We label this path as $P = n_0, f_1, n_1, \ldots, n_{r-1}, f_r, n_r$, and thus $e_j = f_\ell$ for some $1 \leq \ell \leq r$. Let G' be the multigraph that results from removing the edge $e_j = f_\ell$ from the multigraph C_{j-1}. Note that the end points of $e_j = f_\ell$ are $n_{\ell-1}$ and n_ℓ. Let B_1 and B_2 be the connected components of G' that contain $n_{\ell-1}$ and n_ℓ, respectively. From Observation 6.2, we know that $G' = B_1 \cup B_2$. Note that $n_0 \in B_1$ and $n_r \in B_2$. Note also that either $C_j = B_1$ or $C_j = B_2$. Thus, since both n_0 and n_r are exterior nodes, we conclude that $C_k = C_j$ contains an exterior node. ∎

OBSERVATION 6.7. *Assume that every connected component of G contains an exterior node. Let A be a connected component of O_t. Let n_A^{int}, and let e_A be the number of interior nodes of A and the number of edges of A, respectively. Then $e_A \geq n_A^{\text{int}}$.*

Proof. Let n_A be the number of nodes of A, and let n_A^{ext} be the number of its exterior nodes. Note that $n_A = n_A^{\text{int}} + n_A^{\text{ext}}$. From Observation 6.3 we have that $e_A \geq n_A - 1$. From Observation 6.6 we have that $n_A^{\text{ext}} \geq 1$. Thus $e_A \geq n_A^{\text{int}}$. ∎

OBSERVATION 6.8. *Assume that every connected component of G contains an exterior node. Let $n_{O_t}^{\text{int}}$ and e_{O_t} be the number of interior nodes and edges of O_t, respectively. Then $e_{O_t} \geq n_{O_t}^{\text{int}}$.*

Proof. The validity of this observation follows from Observation 6.7. More precisely, $e_{O_t} = \sum e_A \geq \sum n_A^{\text{int}} = n_{O_t}$, where the sums are over all connected components A of O_t, and, as in the previous observation, n_A^{int} and e_A are the number of interior nodes of A and the number of edges of A, respectively. ∎

OBSERVATION 6.9. *Let $n_{O_t}^{\text{int}}$ and n_G^{int} be the number of interior nodes of O_t and G, respectively. Then, $n_{O_t}^{\text{int}} = n_G^{\text{int}}$.*

Proof. We will show that, for each k $(1 \leq k \leq t)$, all the interior nodes of O_{k-1} belong to O_k, and thus all the interior nodes of $G = O_0$ belong to O_t. This, together with the fact that O_t is a submultigraph of G, implies that $n_{O_t}^{\text{int}} = n_G^{\text{int}}$.

We now proceed to show that all the interior nodes of O_{k-1} belong to O_k. Let n be a node that belongs to O_{k-1} but not to O_k. Thus, n is left isolated once e_k is removed from O_{k-1}. This implies that n is an end point of e. On the other hand, as noted in Observation 6.5, e_k belongs to a percolating path P that is included in O_{k-1}. Thus, any node that is an end point of e_k, and is also an interior point, is not an end point of P and thus is not left isolated once e_k is removed from O_{k-1}. As a consequence, we conclude that n should be an exterior point, which completes the proof. ∎

Proof of Theorem 5.2. From Observations 6.8 and 6.9 we have

$$\#\{\text{open edges}\} = e_{O_t} \geq n_{O_t}^{\text{int}} = n_G^{\text{int}} = \#\{\text{interior nodes of } G\}. \tag{6.2}$$

Thus,

$$\begin{aligned} \#\{\text{clogged edges}\} &= \#\{\text{edges of } G\} - \#\{\text{open edges}\} \\ &\leq \#\{\text{edges of } G\} - \#\{\text{interior nodes of } G\}, \end{aligned} \tag{6.3}$$

which proves Theorem 5.2. ∎

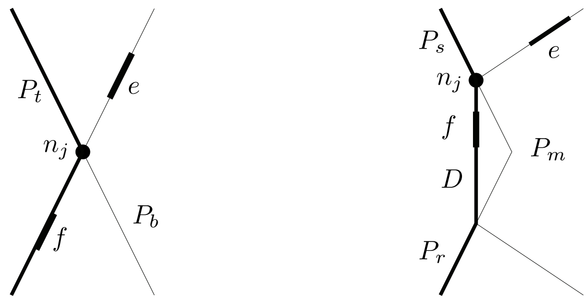

FIG. 5. *Illustration of the arguments in the proof of Observation* 7.1. *The edges* e *and* f *are in very thick lines. In both figures, the percolating path* $\bar{P}$ *is in thick lines.*

7. Optimality of the bound. In this section we will show that our bound (equation (5.1)) is optimal in the following sense.

THEOREM 7.1. *Let* G *be the multigraph that corresponds to one of our filters. Assume that every edge in* G *belongs to a percolating path. Let* $N = \#\{\text{edges of } G\} - \#\{\text{interior nodes of } G\}$. *Then, there is a sequence of edges* $e_1, e_2, \ldots, e_N$ *such that, for each* $1 \le i \le N$, *there is flow through* e_i *when the edges* $e_1, \ldots, e_{i-1}$ *are clogged and all the other edges in* G *are open. In other words, there is a sequence that may clog of length* $N = \#\{\text{edges of } G\} - \#\{\text{interior nodes of } G\}$ *(see Definition* 5.1*).*

7.1. Observations leading to the proof of Theorem 7.1. We will now make some analysis that will lead to the proof of Theorem 7.1.

OBSERVATION 7.1. *Let* G *be the multigraph of one of our filters. Let* f *and* e *be two edges in* G. *Assume that there is a percolating path* P *of the form* $P = A, f, n_0, e_1, \ldots, n_r, e, D$, *where* A *and* D *are paths. Assume also that there is a second percolating path* P' *and* j $(0 \le j \le r)$ *such that* e *does not belong to* P', n_j *belongs to* P', *and none of the nodes* n_i *for* $0 \le i < j$ *belong to* P'. *Then, there is a percolating path* $\bar{P}$ *that contains* f *and does not contain* e.

Proof. If f belongs to P', set $\bar{P} = P'$ and we are done. Thus, assume in the rest of the proof that f does not belong to P'.

Assume first that A does not intersect P'. Then, since the nodes n_i for $0 \le i < j$ do not belong to P', we have that the intersection of $A, f, n_0, e_1, \ldots, n_j$ with P' is only the node n_j. The node n_j splits P' into two paths, P_b and P_t, where P_b connects the bottom boundary with n_j and P_t connects the top boundary with n_j (see the left figure of Figure 5). If A contains a node in the top boundary, we set $\bar{P}$ to be the path $\bar{P} = A, f, n_0, e_1, \ldots, e_j, P_b$. If A contains a node in the bottom boundary, we set $\bar{P}$ to be the path $\bar{P} = A, f, n_0, e_1, \ldots, e_j, P_t$. In either case we have that $\bar{P}$ is a percolating path that contains f and does not contain e (see the left figure in Figure 5).

Assume now that A intersects P'. Let D be the subpath of $A, f, n_0, e_1, \ldots, n_j$ that contains $f, n_0, e_1, \ldots, n_j$, and the only nodes of D that also belong to P' are the end points of D. Let r and s be the end points of D (note that one of these nodes is n_j). The nodes r and s split P' into three paths: P_r, that connects a boundary with the node r; P_s, that connects the other boundary with the node s; and P_m, that connects r and s. Since e does not belong to either D or P', the path $\bar{P} = P_r, D, P_s$ is a percolating path that contains f and does not contain e (see the right figure in Figure 5). This concludes the proof of our observation. ∎

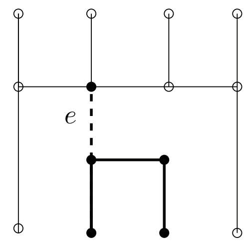

FIG. 6. *Example illustrating the definition of B_e. The edge e is the dashed line, and the edges of B_e are the edge e and the edges shown by the thick solid lines. The nodes of B_e are the black solid circles.*

DEFINITION 7.2. *Let G be the multigraph of one of our filters. Let e be an edge of G. We denote by B_e the submultigraph of G defined by*

$$\{\text{edges of } B_e\} = \{f \text{ edge of } G : f \text{ is included in a percolating path and every percolating path that contains } f \text{ also contains } e\}. \tag{7.1}$$

The nodes of B_e are the end points of the edges of B_e.

An example illustrating the above definition is shown in Figure 6.

OBSERVATION 7.2. *Let G be the multigraph of one of our filters, and let e be an edge of G. Assume B_e is not empty. Let f be an edge of B_e different than e. Let P be a percolating path in G that contains f, and thus P is of the form $P = A, f, n_0, e_1, \ldots, n_r, e, C$. Then, the path $n_0, e_1, \ldots, n_r$ is in fact a path in B_e. In particular, B_e is connected.*

Proof. Our proof is by contradiction. Assume there exists i such that $1 \le i < r$ and e_i is not an edge of B_e. Then, there is a percolating path P' that does not contain e and contains e_i. Let j be the smallest nonnegative integer such that n_j belongs to P'. Note that $0 \le j \le i-1$. We are now in the conditions of Observation 7.1. This implies that there is a percolating path $\bar{P}$ that contains f but does not contain e. This is a contradiction because f belongs to B_e. Thus, the observation is proved. ∎

OBSERVATION 7.3. *Let G be the multigraph of one of our filters, and let e be an edge of G. If f is an edge of B_e, then B_f is a submultigraph of B_e.*

Proof. Let g be an edge of B_f. Then, on the one hand, g is included in a percolating path P'. On the other hand, if P is a percolating path that contains g, then P contains f. Since f is an edge of B_e, P also contains e. Thus, g is an edge of B_e. ∎

OBSERVATION 7.4. *Let G be the multigraph of one of our filters, and let a be one of its interior nodes. Let e be an edge that has a as one of its end points. Let b be the other end point of e. Let p_a and p_b be the pressures at a and b, respectively. If $p_a \ne p_b$, there exists an edge f such that* (1) *a is an end point of f,* (2) *f is not a loop, and* (3) *if c is the other end point of f and p_c is the pressure at the node c, then either $p_c < p_a < p_b$ or $p_b < p_a < p_c$.*

Proof. If $p_a < p_b$, fluid flows through e from the node b into the node a. Since a is an interior node, mass conservation implies that fluid flows out of a through another edge f that has a as an end point. If c is the other end point of f, we have that $p_a > p_c$ because fluid flows from a to c. This proves the observation in the case $p_a < p_b$. The case $p_a > p_b$ is proved analogously. ∎

OBSERVATION 7.5. *Let G be the multigraph of one of our filters, and let e be an edge of G. Assume that there is flow through e. If B_e is not a path, then there are three edges of B_e such that they meet at a common end point and there is flow through those three edges.*

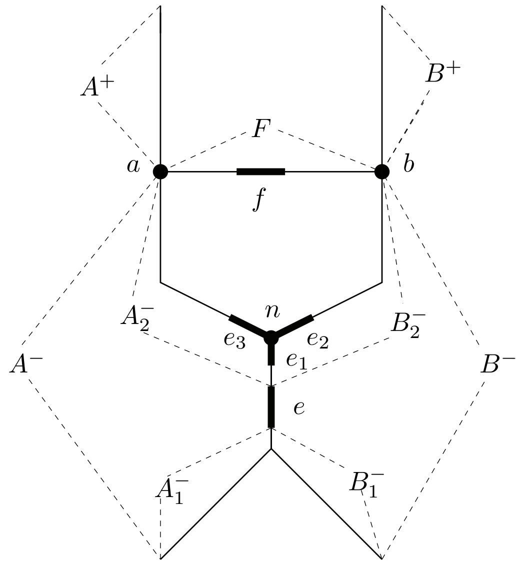

FIG. 7. *Part of the multigraph (in solid lines) illustrating the arguments in the proof of Observation* 7.5. *The dashed lines are not part of the multigraph. They connect the end points of the paths with the corresponding path names. The thick short segments are edges. The node n is the intersection of three edges of B_e, e_1, e_2, and e_3, and there is flow through those three edges.*

Proof. Since there is flow through e, e belongs to a percolating path (see Observation 6.4). Thus, e belongs to B_e, implying that B_e is not empty. Assume B_e is not a path. If there is flow through all the edges in B_e, the observation is true because B_e is not a path, and thus, three of its edges must meet at a node.

Assume now that there is an edge f in B_e such that there is no flow through f. Note that the pressure is equal at both ends of f. We denote this pressure by p_f. Let P be a percolating path that contains f. Such a path exists because f belongs to B_e. Let F be the largest subpath of P that contains f, and the pressure is equal to p_f at all the nodes of F. Recall that the pressures at the top and bottom boundaries are p_t and p_b, respectively, with $p_b > p_t$. Using the facts that F contains at least one edge, F is a subpath P, and P is a percolating path, it can be shown that $p_t < p_f < p_b$.

Let a and b be the end points of F. With the help of Observation 7.4, it can be shown that there exists a path A^- in G that connects the bottom boundary with a such that the pressure decreases from one node to the next one as we walk along A^- from the bottom boundary toward a. Analogously, there exist paths A^+, B^-, and B^+ in G such that (1) A^+ connects a with the top boundary, and the pressure decreases from one node to the next one as we walk along A^+ from a to the top boundary; (2) B^- connects the bottom boundary with b, and the pressure decreases from one node to the next one as we walk along B^- from the bottom boundary to b; and (3) B^+ connects b with the top boundary, and the pressure decreases from one node to the next as we walk along B^+ from b to the top boundary (see Figure 7).

We first observe that there is flow through all the edges in A^-, A^+, B^-, and B^+. Note also that A^-, F, B^+ and B^-, F, A^+ are two different percolating paths that contain the edge f. Since f is an edge of B_e, we also have that e is contained in both paths A^-, F, B^+ and B^-, F, A^+. Note that there is no flow through any edge of F, and thus, since there is flow through e, e is not an edge of F. Assume that e

belongs to A^-. This implies that the pressure at each end point of e is greater than or equal to p_f, and thus, since e also belongs to B^-, F, A^+, e should belong to B^-. As a consequence, the pressure at each end point of e is in fact strictly greater than p_f (see Figure 7).

Thus, the path A^- is of the form $A^- = A_1^-, e, A_2^-$, with a being an end point of A_2^-, and B^- is of the form $B^- = B_1^-, e, B_2^-$, with b being an end point of B_2^- (see Figure 7). From Observation 7.2, both A_2^- and B_2^- are included in B_e. Since the pressure at the end points of e is strictly greater than p_f, each path A_2^- and B_2^- has at least one edge. Since A_2^- and B_2^- share only one end point, as we walk from that common end point (which is one of the end points of e) in the flow direction, A_2^- and B_2^- eventually branch out at some node that we call n. That node n is the intersection of three edges in B_e, e_1, e_2, and e_3, and there is flow through those three edges (see Figure 7). ◻

OBSERVATION 7.6. *Let G be the multigraph of one of our filters, and let e be an edge of G. Assume that there is flow through e. If B_e is not a path, then there exists f, an edge of B_e, such that B_f is not empty, $B_f \neq B_e$, and there is flow through f.*

Proof. From Observation 7.5, there exist three edges e_1, e_2, and e_3 of B_e that have a common end point, and there is flow through all three of these edges. Let P be a percolating path in G that contains e_1. Every node in a path has at most degree 2. Thus, at least one of the edges e_2 or e_3 does not belong to P. Assume e_3 does not belong to P. Then, e_1 does not belong to B_{e_3} and thus, $f = e_3$ is an edge of B_e such that $B_f \neq B_e$, B_f is not empty, and there is flow through f. ◻

OBSERVATION 7.7. *Let G be the multigraph of one of our filters. If there exists a percolating path in G, then there is an edge e of G such that there is flow through e, B_e is not empty, and B_e is a path.*

Proof. It can be easily shown that, since there exists a percolating path in G, there is flow through at least an edge in G. Select an edge e' in G such that there is flow through e'. It can be easily shown that $B_{e'}$ is not empty. If $B_{e'}$ is a path, set $e = e'$, and we are done. Otherwise, from Observation 7.6 there exists an edge e such that B_e is a submultigraph of $B_{e'}$, $B_e \neq B_{e'}$, there is flow through e, and B_e is not empty. Since our multigraphs are finite, we cannot continue this procedure an infinite number of times. Thus, we are eventually left with an edge e such that B_e is a nonempty path and there is flow through e. ◻

OBSERVATION 7.8. *Let G be the multigraph of one of our filters. Let e be an edge of G such that B_e is a path, $B_e = n_0, e_1, n_1, \ldots, e_r, n_r$, for some $r \geq 1$. Then, for each $1 \leq i < r$, n_i is an interior node of G.*

Proof. Assume that there is $1 \leq i < r$ such that n_i is an exterior node, i.e., n_i belongs to one of the boundaries, say the bottom one. Since $e \in B_e$, there exist $1 \leq \ell \leq r$ such that $e = e_\ell$. Assume that $i < \ell$.

Let P be a percolating path that contains e_1. Since e_1 belongs to B_e, we have that e is an edge of P. Thus, P is of the form A, e_1, C, e, D, where A, C, and D are disjoint paths, A and D each contain a node in different boundaries (one of its end points), and C does not contain any node in the boundaries. Due to Observation 7.2, C is contained in B_e. Since B_e is the path $B_e = n_0, e_1, n_1, \ldots, e_r, n_r$, the facts that the path C connects e_1 with $e = e_\ell$ and is included in B_e imply that $C = n_1, \ldots, e_{\ell-1}, n_{\ell-1}$. Thus, since $1 \leq i < \ell$, we have that the exterior node n_i belongs to C, which is a contradiction. Thus, for each $1 \leq i < \ell$, n_i is an interior node of G. The case $\ell \leq i$ is proved analogously. ◻

OBSERVATION 7.9. *Let G be the multigraph of one of our filters. Assume that every edge in G belongs to a percolating path. Let e be an edge of G such that B_e is a* *path, $B_e = n_0, e_1, n_1, \dots, e_r, n_r$. Then, for each $1 \le i < r$, the degree of n_i as a node of G is 2.*

Proof. Since $e \in B_e$, there exists $1 \le \ell \le r$ such that $e = e_\ell$.

Let i be the smallest number that satisfies $\ell \le i < r$, and the degree of n_i as a node of G is at least 3. Assume that such i exists. Note that, from Observation 7.8, n_i is an interior node. Let f be an edge that is not in B_e that has n_i as an end point. Given the hypothesis of the observation, there is a percolating path P' that contains f and does not contain e. Note that P' contains n_i. Note also that n_s is not in P' for any $\ell \le s < i$.

Let P be a percolating path that contains e_{i+1}. Since e_{i+1} is in B_e, the path P contains $e_\ell = e$. Thus, P is of the form $P = D, e_\ell, F, e_{i+1}, E$. From Observation 7.2, F is included in B_e. This last fact, together with the fact that B_e is the path $B_e = n_0, e_1, n_1, \dots, e_r, n_r$, implies that $P = D, e_\ell, n_\ell, \dots, n_i, e_{i+1}, E$, where the path D connects one boundary with $n_{\ell-1}$, the path E connects the other boundary with n_{i+1}, each of the paths D and E has only one node in the boundaries, and neither path contains $e = e_\ell$.

Since P' contains n_i and does not contain e, we can apply Observation 7.1 (with e_{i+1} here playing the role of f in Observation 7.1) to conclude that there is a percolating path $\bar{P}$ that contains e_{i+1} and does not contain e. This is a contradiction because e_{i+1} belongs to B_e, and thus the observation is proved. ∎

OBSERVATION 7.10. *Let G be the multigraph of one of our filters. Assume that every edge in G belongs to a percolating path in G. Let e be an edge of G such that B_e is a nonempty path. Let G' be the multigraph that results from first removing from G the edges that are in B_e and then removing any node that is left isolated, i.e.,*

$$\{\textit{edges of } G'\} = \{\textit{edges of } G\} - \{\textit{edges of } B_e\}, \tag{7.2}$$

and the nodes of G' are the end points of the edges in G'.

Then, if G' is not empty, every edge in G' belongs to a percolating path in G' and

$$\begin{aligned} &\#\{\textit{edges of } G'\} - \#\{\textit{interior nodes of } G'\} \\ &\quad = \#\{\textit{edges of } G\} - \#\{\textit{interior nodes of } G\} - 1. \end{aligned} \tag{7.3}$$

Proof. If e' is an edge in G', then e' is not an edge of B_e, and thus there exists a percolating path P that contains e' and does not contain e. As a consequence, none of the edges in P belong to B_e. Thus, P is in fact a path in G'.

Since B_e is a path, B_e is of the form $B_e = n_0, e_1, n_1, \dots, e_r, n_r$, where, as shown in the previous observations, the nodes $n_1, n_2, \dots, n_{r-1}$ are interior nodes and each of these nodes has degree 2 as a node of G. Thus, once the edges $e_1, e_2, \dots, e_r$ are removed from G to obtain G', all the nodes $n_1, n_2, \dots, n_{r-1}$ are left isolated and thus also removed from G. If n_0 is an interior node, it is easy to see that n_0 is not left isolated once $e_1, e_2, \dots, e_r$ are removed. The same applies for n_r. In other words, $n_1, n_2, \dots, n_{r-1}$ are the only interior nodes that are left isolated once the edges in B_e are removed. Thus, we have

$$\#\{\text{edges of } G'\} = \#\{\text{edges of } G\} - r \tag{7.4}$$

and

$$\#\{\text{interior nodes of } G'\} = \#\{\text{interior nodes of } G\} - (r-1), \tag{7.5}$$

from whence the validity of (7.3) follows. ∎

Observation 7.11. *Let G be the multigraph of one of our filters and let e be an edge in G. Assume that there is flow through e. Let G' be the same multigraph as in Observation* 7.10*, i.e., G' results from first removing from G the edges that are in B_e and then removing any node that is left isolated* (*see* (7.2)).

Then, the flow in G when e is the only edge clogged is the same as the flow in G' when none of the edges in G' are clogged.

This observation does not require proof. It is a consequence of the definition of B_e and the fact that there is no flow through clogged edges.

7.2. Proof of Theorem 7.1. We prove this theorem by induction on $N = \#\{\text{edges of } G\} - \#\{\text{interior nodes of } G\}$.

If $N = 1$, then G is a path. Thus, there is flow through all the edges in G. As a consequence, we can select e_1 to be any edge, and the sequence e_1 is a sequence that may clog of length 1, which proves the case $N = 1$.

Assume now that $N > 1$. From Observation 7.7, we select e_1 such that there is flow through e_1 and B_{e_1} is a path. Let G_1 be the multigraph that results from first removing from G the edges that are in B_{e_1} and then removing any node that is left isolated. Let $N_1 = \#\{\text{edges of } G_1\} - \#\{\text{interior nodes of } G_1\}$. From Observation 7.10, we have that $N_1 = N - 1$. Thus, by an inductive hypothesis, there exist $e_2, e_3, \ldots, e_N$ as a sequence of edges that may clog in G_1 of length $N - 1$. Finally, from Observation 7.11, we have that in fact, $e_1, e_2, \ldots, e_N$ is a sequence of edges that may clog in G of length N, which concludes the proof of the theorem.

8. Asymptotic formula when the edges are much shorter than the distance between the boundaries. As always, G is the multigraph of one of our filters. We recall that d_G, the average degree of G, satisfies $d_G = 2e_G/n_G$, where e_G and n_G are the numbers of edges and nodes of G, respectively (see (3.1) and section 3).

Assume that $\#\{\text{exterior nodes of } G\} \ll \#\{\text{interior nodes of } G\}$. This is the case when the length of the edges is much smaller than the distance between the boundaries.

Assume also that $\#\{\text{edges of } G\} - \#\{\text{nodes of } G\} \gg \#\{\text{exterior nodes of } G\}$. This will usually be the case. (This is not satisfied when G does not differ much from a graph consisting of the union of disjoint percolating paths.)

In the parameter regime of the above assumptions, we have that

$$\begin{aligned}&\#\{\text{edges of } G\} - \#\{\text{interior nodes of } G\}\\ &\quad\simeq \#\{\text{edges of } G\} - \#\{\text{nodes of } G\} = \frac{d_G - 2}{d_G}\#\{\text{edges of } G\}.\end{aligned} \tag{8.1}$$

Thus, our bound, Theorem 5.2, reads

$$\#\{\text{clogged edges}\} \lesssim \frac{d_G - 2}{d_G}\#\{\text{edges of } G\}. \tag{8.2}$$

9. A three-dimensional example. We now consider a three-dimensional example. Assume that our filter is a bounded section of a cubic lattice with two of the axes being parallel to the boundaries of the filter. Then, the degree of most interior nodes is 6. Thus, if the length of the edges is much smaller than the distance between the boundaries, and if the number of exterior nodes is much bigger than one, our approximation of (8.2) is valid and becomes

$$\#\{\text{clogged edges}\} \lesssim \frac{2}{3}\#\{\text{edges of } G\}. \tag{9.1}$$

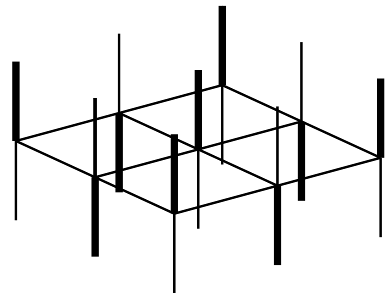

FIG. 8. *In this example $H = W = 2$. The thick lines are the thick edges and the thin lines the thin edges.*

Note that, in the example of this section, every edge is included in a percolating path. We have shown that, for this kind of filter, our bound is sharp; i.e., there is a sequence of edges that may clog whose length is equal to our bound, and thus it has the asymptotic value of the right-hand side of inequality (9.1). However, as previously discussed, there are many sequences of edges that may clog and make the filter nonpermeable, and the length of most of them is less than our bound. Thus, the number of edges that actually clog and make the filter nonpermeable may be smaller than our bound.

The physical mechanisms that lead to the clogging of channels may be complex and depend on the particular problem under consideration. Nevertheless, for illustrative purposes, let us assume here the following simple rules. Each channel is either thin or thick. Thick channels never clog and thin channels eventually clog if there is flow through them. We ask the following question for the example we are considering: *How should we select the width of the channels so that the number of channels that actually clog is equal to our bound?*

To answer the above question, we first specify the location and length of the edges. Assume all the edges have length 1, the bottom boundary is located at $z = 0$, and the top boundary at $z = H$, where H is a large positive integer. The vertical edges form the lines $Z_{ij} = \{x = i,\ y = j, \text{ and } 0 \le z \le H\}$, where $0 \le i, j \le W$. Thus, the horizontal edges form the lines $X_{jk} = \{y = j,\ z = k, \text{ and } 0 \le x \le W\}$, where $0 \le j \le W$ and $1 \le k < H$, and $Y_{ik} = \{x = i,\ z = k \text{ and } 0 \le y \le W\}$, where $0 \le i \le W$ and $1 \le k < H$ (see Figure 8).

We now select the thin and thick edges. All the edges parallel to the x or y axis are selected thin. The lowest edges in the lines Z_{ij} with $i + j$ even are selected thin, and the highest edges in the lines Z_{ij} with $i + j$ odd are selected thin. All other vertical edges are selected thick. In other words,

$$\begin{array}{l}\text{thickness of edge} \\ \{x = i,\ y = j,\ (k-1) \le z \le k\}\end{array} = \begin{cases} \text{thin} & \text{if } i + j \text{ is even and } k = 1, \\ \text{thin} & \text{if } i + j \text{ is odd and } k = H, \\ \text{thick} & \text{otherwise.} \end{cases} \tag{9.2}$$

It is easy to verify that, as long as there is a thin edge open, there is flow through the filter and through at least a thin open edge. Thus, eventually all the thin edges will clog. At that point, there is no more flow through the filter.

It is also easy to check that

$$\begin{aligned} \#\{\text{thick edges}\} &= (W+1)^2(H-1), \\ \#\{\text{thin edges}\} &= (W+1)^2 + 2(H-1)(W+1)W, \\ \#\{\text{interior nodes}\} &= (W+1)^2(H-1). \end{aligned} \tag{9.3}$$

Thus, it is clear that by the time all the thin edges clog, we have

$$\#\{\text{clogged edges}\} = \#\{\text{edges}\} - \#\{\text{interior nodes}\}, \tag{9.4}$$

which shows that our upper bound (see Theorem 5.2) is realized. Note also that, since $H \gg 1$ and $W \gg 1$, we also have

$$\frac{\#\{\text{clogged edges}\}}{\#\{\text{edges}\}} \approx \frac{2}{3}, \tag{9.5}$$

which shows that the asymptotic bound (8.2) is realized.

10. An alternative formula for two-dimensional filters.

10.1. Review: Planar multigraph and Euler's formula. A multigraph is *planar* if it can be drawn in the plane in such a way that any two different edges may only intersect at one or two of their end points. Any such drawing is a *plane drawing* of the multigraph. The multigraphs of two-dimensional filters are planar multigraphs. We identify each planar multigraph with one of its plane drawings. In the rest of this section, any multigraph that we mention or consider is a planar multigraph.

A multigraph divides the plane into regions called *faces*. More precisely, the faces are the connected components of what is left from the plane once we remove the multigraph from the plane. In other words, the faces are the connected components of the set of points in the plane that do not belong to any edge of the multigraph. Note that the faces are open sets. Any finite multigraph has one unbounded face surrounding it, called the *infinity face*.

Let G be a multigraph. We denote by n_G its number of nodes, by e_G its number of edges, and by f_G its number of faces. The well-known *Euler formula* states that, if G is connected,

$$n_G + f_G = e_G + 2. \tag{10.1}$$

10.2. Alternative formula for two-dimensional filters. We will now obtain an equivalent formula of our bound (Theorem 5.2) that is valid for two-dimensional filters. Let G be the multigraph of one of our filters. We denote by $\bar{G}$ the multigraph that results from adding to G edges in the boundaries connecting the exterior nodes. More precisely, $\bar{G}$ and G have the same nodes, $\bar{G}$ contains all the edges in G, and $\bar{G}$ also contains the edges of the path in $z = z_b$ connecting the leftmost node in the bottom boundary with the rightmost node in that boundary, and also the edges of the path in $z = z_t$ connecting the leftmost node in the top boundary with the rightmost node in that boundary. An example is given in Figure 9.

Observation 10.1.

$$\#\{\textit{edges of } \bar{G}\} - \#\{\textit{edges of } G\} = \#\{\textit{exterior nodes of } G\} - 2. \tag{10.2}$$

Proof. This observation should be clear. The number of edges of $\bar{G}$ in the top boundary equals the number of nodes of G in the top boundary minus one. The

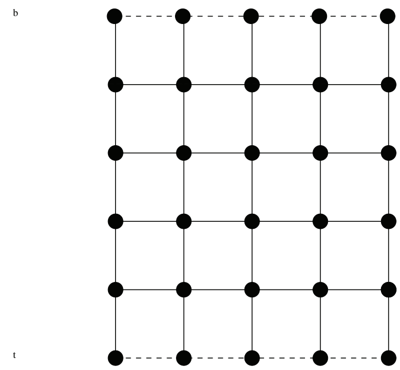

FIG. 9. *Example illustrating the definition of $\bar{G}$. The edges of G are the solid lines. The edges of $\bar{G}$ are both the solid and the dashed lines.*

same is valid in the bottom boundary. These facts imply the validity of this observation. ∎

OBSERVATION 10.2.

$$\begin{aligned}&\#\{\textit{edges of } G\} - \#\{\textit{interior nodes of } G\}\\ &\quad = \#\{\textit{edges of } \bar{G}\} - \#\{\textit{nodes of } G\} + 2.\end{aligned} \tag{10.3}$$

Proof. This observation follows from Observation 10.1. ∎

OBSERVATION 10.3.

$$\#\{\textit{edges of } G\} - \#\{\textit{interior nodes of } G\} = \#\{\textit{faces of } \bar{G}\}. \tag{10.4}$$

Proof. This observation follows from Observation 10.2 and Euler's formula. ∎

Thus, an alternative formulation of our Theorem 5.2 is as follows.

THEOREM 10.1. *Let G be the multigraph that corresponds to one of our two-dimensional filters, and let $\bar{G}$ be the multigraph that results from connecting the exterior nodes of G with edges included in the boundaries. Then,*

$$\#\{\textit{clogged edges}\} \leq \#\{\textit{faces of } \bar{G}\}. \tag{10.5}$$

This two-dimensional result was previously obtained in [18, 19] by other means.

11. Final discussion. Our work was motivated by the following questions: (1) *How much pore space is used to trap particles in a filter?* (*Or alternatively, how much pore space is left clean, i.e., without particles, in a filter once it becomes impermeable?*) (2) *How does this amount of unused pore space depend on the micro-geometry, i.e., shape and geometry of the pore space?*

In an attempt to shed light on the answers to these important questions, we introduced a network model and obtained a bound on the number of channels that clog (Theorem 5.2). This bound is given in terms of the properties of the network and is sharp (in the sense explained in section 7). These results are novel and provide new insight into the relation between pore geometry and filter efficiency in an idealized context. We hope that the lessons learned will eventually translate into novel ideas in the design of real filters.

While we have discussed only clogging of networks, our analysis and results apply to other physical problems. Discrete models of electrical breakdown consist of networks of conducting bonds, where each bond breaks down and becomes nonconducting when the current within that bond exceeds a certain critical value. In this context, our results provide an upper bound on the number of bonds that break down and become nonconducting. On the other hand, in the context of brittle networks, we provide an upper bound on the number of bonds that break. We refer the reader to the book [30] and the references therein for more detailed discussions of electrical breakdown and brittle failure.

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