SIAM J. APPL. MATH.
Vol. 66, No. 1, pp. 228–236

VOLUME OF SUSPENSION THAT FLOWS THROUGH A SMALL ORIFICE BEFORE IT CLOGS*

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Abstract. We consider the following experiment. A container is filled with a suspension consisting of particles immersed in an incompressible liquid. An opening is made on the container wall and the suspension flows through the opening. We develop a mathematical model to compute the expected volume of suspension extracted before particles clog the opening. Our studies are relevant to the understanding of clogging of pore throats in porous media, which plays an important role in geomaterials, biological systems, and industrial applications.

Key words. clogging, suspension flow, porous media, mathematical modeling

AMS subject classifications. 76S05, 76M99

DOI. 10.1137/040616164

1. Introduction.

The migration of fines, i.e., small particles, in porous media plays an important role in several engineering applications including oil production, soil erosion, ground water pollution, and the operation of filter beds. Accordingly, this topic is an active area of research in a number of disciplines including petroleum, geotechnical, chemical, environmental, and hydraulic engineering (see [6]).

Soil mass is an example of a porous medium. The particles that hold the material together form what is known as the load carrying skeleton. Fines are small particles that do not form part of the load-carrying skeleton. Rocks are other examples of porous media with fines present in them. A typical size of these fines, which can be of inorganic, organic, or biological nature, is 1 μm, and they may have an electric surface charge. If liquid flows through the porous medium, fines attached to pore surfaces may be released due to hydrodynamic forces. These fines will move with the flow and be retained at other locations or exit the porous medium. The sites that retain fines are usually pore constrictions or pore throats. If several migrating particles reach a small pore throat simultaneously, the particles may clog the pore throat. More detailed discussions on the physical phenomena that lead to clogging can be found in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

If fines get captured, the porous medium may become plugged. On the other hand, when fines exit the medium, the porous medium may erode, which may result in structural failure. Examples where these phenomena have important consequences include the following: the extraction of petroleum, where plugging is an undesirable effect—if the well completely clogs, it can no longer be used; the containment of contaminants—plugging may help in this situation; the failure of earthen dams and roads, which can be caused by the erosion that results from particle migration.

In this paper we study the following simple experiment that models aspects of clogging at a single pore throat. A container is filled with a suspension made of an incompressible liquid and spherical particles. A circular opening is made in the container wall through which the suspension flows. The particles may or may not clog

*Received by the editors October 1, 2004; accepted for publication (in revised form) May 16, 2005; published electronically November 4, 2005. This research was supported by the NSF.
http://www.siam.org/journals/siap/66-1/61616.html

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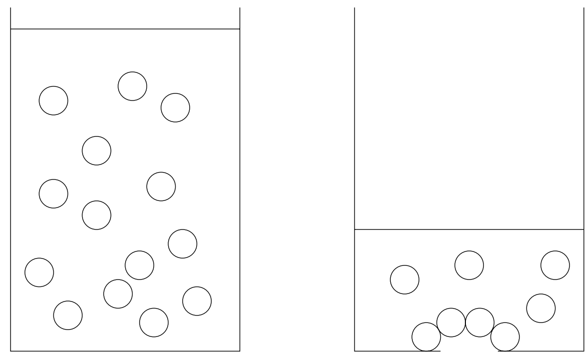

FIG. 1.1. *Suspension in a container. The left-hand image shows the container filled with the suspension before the opening is made. The right-hand image shows the system at the moment the opening clogs.*

the opening. Our goal is to predict the volume of fluid extracted before clogging (if clogging does occur). The experiment described is illustrated in Figure 1.1.

The mathematical modeling of migration of fines in porous media is a complex task that is in its infancy (see [6]). The objective of this paper is to provide a step toward the more ambitious goal of developing reliable models for studying more complex problems where migration of fines in porous media plays an important role.

The rest of the paper proceeds as follows. In section 2 we make our physical assumptions and describe our mathematical model. In section 3 we describe a numerical algorithm to obtain solutions of the model. In section 4 we derive an upper bound on the volume extracted before clogging, and in section 5 we obtain a lower bound. The paper ends in section 6 with examples and conclusions.

2. The model. Our model relies on the following approximations. The liquid is incompressible. The flow is not disturbed by the presence of particles. The center of each particle flows with the same velocity as the fluid. Before the opening is made, the center of each particle is randomly placed inside the container with a uniform probability distribution in space.

Note that the initial location of the centers of the particles are independent random variables, and thus we allow particles to overlap.

For each point x in the container, we denote by $F(x)$ the volume of fluid extracted by the time the element of fluid initially at x reaches the opening. The left-hand image in Figure 2.1 shows a two-dimensional sketch of level sets of the function F. (The actual level sets of F are surfaces within the three-dimensional container.) Due to the incompressibility of the fluid, the region enclosed by the level sets $\{x : F(x) = V + \Delta V\}$ and $\{x : F(x) = V\}$ has volume ΔV.

We denote by A the area of the orifice and by v the volume fraction of particles (i.e., the volume occupied by the particles divided by the volume of the suspension). All the particles have the same radius r.

To motivate our criteria for clogging, assume that the fluid velocity is constant in space across the opening and out of the container. Once the volume of suspension initially in $\{x : V < F(x) \leq V + rA\}$ leaves the container, it forms a cylinder with height r (see the right-hand image in Figure 2.1). Since the centers of particles flow with the fluid, the number of centers of particles that belong to this cylinder is equal to the number of centers of particles initially placed in $\{x : V < F(x) \leq V + rA\}$. We

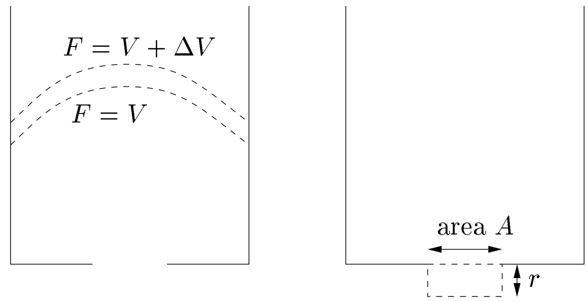

FIG. 2.1. *The left-hand image is a two-dimensional sketch of level sets of F. The region enclosed by the dashed lines is $\{x : V < F(x) \leq V + \Delta V\}$. The right-hand image is a two-dimensional sketch of the three-dimensional cylinder (enclosed by dashed lines) which is formed by the suspension initially in $\{x : V < F(x) \leq V + rA\}$ as soon as it leaves the container.*

denote this number by $k(V)$, i.e.,

(2.1) $$k(V) = \text{ number of particles initially placed in } \{x : V < F(x) \leq V + rA\}.$$

Note that the particles whose centers belong to the dashed cylinder of Figure 2.1 arrive *almost simultaneously* at the opening. Thus, we propose that clogging occurs when $k(V)$, the number of particles arriving almost simultaneously at the opening, exceeds a threshold $k_{\max}$ for the first time. Thus, if the opening clogs, the volume of fluid that is extracted before clogging is

(2.2) $$V^\star = \min_{\{V : V \geq 0 \text{ and } k(V) > k_{\max}\}} V.$$

We define λ to be the ratio of the volume of the dashed cylinder of Figure 2.1 and the volume of a particle, i.e.,

(2.3) $$\lambda = \frac{3A}{4\pi r^2}.$$

Since the number of centers of particles that can belong to the cylinder under the condition that the particles do not overlap increases linearly with λ, we assume that $k_{\max}$ is of the form

(2.4) $$k_{\max} = \gamma \lambda,$$

where γ is a material parameter to be experimentally determined. Given a realization of initial distribution of centers of particles inside the container, (2.1)–(2.4) determine the extracted volume $V^\star$.

3. Algorithm to compute the extracted volume. Assume that the suspension has volume $\mathcal{V}$ and contains a large but finite number N of particles. Then, the volume fraction of particles is $v = N 4\pi r^3/(3\mathcal{V})$. The initial location of the center of each particle is a random variable with uniform probability distribution. This fact along with the incompressibility of the fluid implies that the volume extracted by the time a center of a particle reaches the opening is also a random variable with uniform probability distribution. As a consequence, if V_i is the volume extracted when the ith particle reaches the opening, these volumes V_i are the result of ordering N

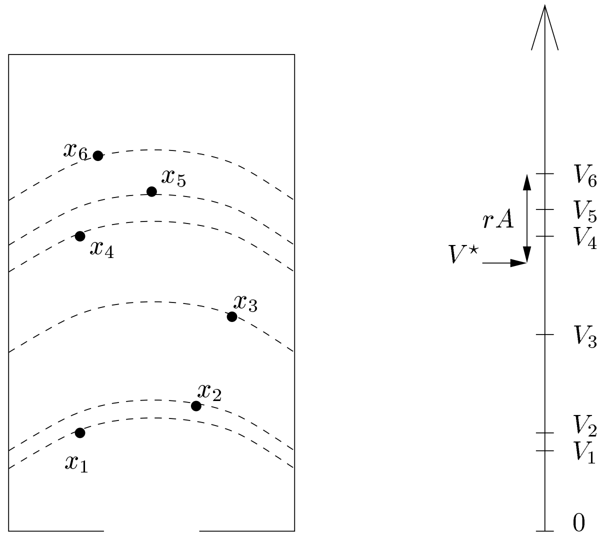

Fig. 3.1. *The initial location of the center of the ith particle to reach the opening is x_i. The dashed lines are the level sets of F. $V_i = F(x_i)$ is the volume extracted when the center of the ith particle reaches the opening. $V^\star$ is the volume extracted before clogging.*

numbers selected independently with uniform probability distribution in the interval $[0, \mathcal{V}]$. This is illustrated in Figure 3.1.

Our criterion for clogging (described in section 2) is illustrated in Figure 3.1. If we place a segment of length rA on top of the vertical volume axis of Figure 3.1 with the left end at 0, then move the segment in the upward direction, and stop as soon as the segment covers more than $k_{\max}$ particles simultaneously, the location of the lower end of the segment is the extracted volume $V^\star$.

The above paragraph can be precisely described as follows. For each i we define n_i to be the largest integer such that $V_{i-n_i+1} > V_i - rA$ subjected to the restriction $n_i \leq i$. If there exists $i \in [1, N]$ such that $n_i > k_{\max}$, clogging occurs. Assuming that this is the case, let $i^\star = \min\{i : n_i > k_{\max}\}$. Then

$$V^\star = \begin{cases} 0 & \text{if } V_{i^\star} < rA, \\ V_{i^\star} - rA & \text{if } V_{i^\star} \geq rA. \end{cases} \tag{3.1}$$

The present discussion leads to the following algorithm to compute $V^\star$ for a given realization:

$i \leftarrow 1$
$n \leftarrow 1$
While $n \leq k_{\max}$ and $i < N$
    $i \leftarrow i + 1$
    $n \leftarrow n + 1$
    While $V_{i-n+1} \leq V_i - rA$
        $n \leftarrow n - 1$
    end
end
If $i = N$ and $n \leq k_{\max}$ then
    "No clogging"

else

$V^\star \leftarrow \max\{0, V_i - rA\}$

end

The expected volume extracted before clogging, $E(V^\star)$, is computed by averaging the values of $V^\star$ obtained for a large number of different realizations. Note that the complexity of this algorithm in $O(N)$.

4. Upper bound on the expected extracted volume of suspension before clogging. Given a realization, the extracted volume before clogging $V^\star$, assuming that clogging does occur, is the minimum of the function $f(V) = V$ over the set $\{V : V \geq 0 \text{ and } k(V) > k_{\max}\}$ (see (2.2)). We define U to be the minimum of the same function $f(V) = V$ over a smaller set. More precisely,

$$U = \min_{\{V : V = irA,\ i \text{ integer},\ i \geq 0,\ k(V) > k_{\max}\}} V. \tag{4.1}$$

Since U and $V^\star$ are the minimum of the same function $f(V) = V$, but the set where f is minimized to obtain U is a subset of the set where f is minimized to obtain $V^\star$, we have

$$V^\star \leq U. \tag{4.2}$$

Thus, $E(V^\star)$ and $E(U)$, the expected values of $V^\star$ and U, respectively, satisfy

$$E(V^\star) \leq E(U). \tag{4.3}$$

In Appendix A we show that, if $rA \ll \mathcal{V}$ ($\mathcal{V}$ is the initial volume of the suspension),

$$E(U) = \frac{\mu}{1-\mu} rA, \quad \text{where} \quad \mu = e^{-\lambda v} \sum_{i=0}^{k_{\max}} \frac{(\lambda v)^i}{i!}, \tag{4.4}$$

where we recall that v is the volume fraction of the particles and λ was defined in (2.3). In particular, we also show in Appendix B that, in the parameter regime $\lambda v \ll 1$, we have

$$E(U) \simeq \frac{([k_{\max}]+1)!}{(\lambda v)^{[k_{\max}]+1}} rA, \tag{4.5}$$

where $[k_{\max}]$ is the integral part of $k_{\max}$, i.e., the largest integer that is not greater than $k_{\max}$.

5. Lower bound on the expected extracted volume of suspension before clogging. Let M be the positive integer that satisfies

$$(M-1)rA \leq V^\star < MrA. \tag{5.1}$$

The sets $\{x : (M-1)rA < F(x) \leq MrA\}$ and $\{x : MrA < F(x) \leq (M+1)rA\}$ are disjoint, and their union contains $\{x : V^\star < F(x) \leq V^\star + rA\}$. Thus, since the number of particle centers initially placed in $\{x : V^\star < F(x) \leq V^\star + rA\}$ is larger than $k_{\max}$, the number of particle centers initially placed in one of the sets $\{x : (M-1)rA < F(x) \leq MrA\}$ or $\{x : MrA < F(x) \leq (M+1)rA\}$ is larger than $k_{\max}/2$. In other words, $\max\{k((M-1)rA), k(MrA)\} > k_{\max}/2$.

We define

$$L = \min_{\{V: V = irA,\ i \text{ integer},\ i \geq 0,\ k(V) > k_{\max}/2\}} V. \tag{5.2}$$

Given the above discussion, we have that $L \leq MrA$. Thus, (5.1) implies

$$L - rA \leq V^{\star} \tag{5.3}$$

and thus

$$E(L) - rA \leq E(V^{\star}), \tag{5.4}$$

where, as in the previous section, $E(.)$ denotes the expected value of the expression between brackets.

Following the same arguments to compute the upper bound, we obtain that, in the regime $rA \ll \mathcal{V}$,

$$E(L) = \frac{\eta}{1-\eta} rA, \quad \text{where} \quad \eta = e^{-\lambda v} \sum_{i=0}^{k_{\max}/2} \frac{(\lambda v)^i}{i!}. \tag{5.5}$$

In particular, in the parameter regime $\lambda v \ll 1$, we have

$$E(L) \simeq \frac{([k_{\max}/2] + 1)!}{(\lambda v)^{[k_{\max}/2]+1}} rA \tag{5.6}$$

(as before [.] is the integral part of the argument).

6. Examples and conclusions. As an illustrative example, in Figure 6.1 we show a plot of the expected extracted volume $E(V^{\star})$ and the upper and lower bounds $E(U)$ and $E(L) - rA$ versus the volume fraction v. The parameter values chosen are $\lambda = 3$ and $\gamma = 1$ (and thus, $k_{\max} = \lambda$). The expected extracted volumes were

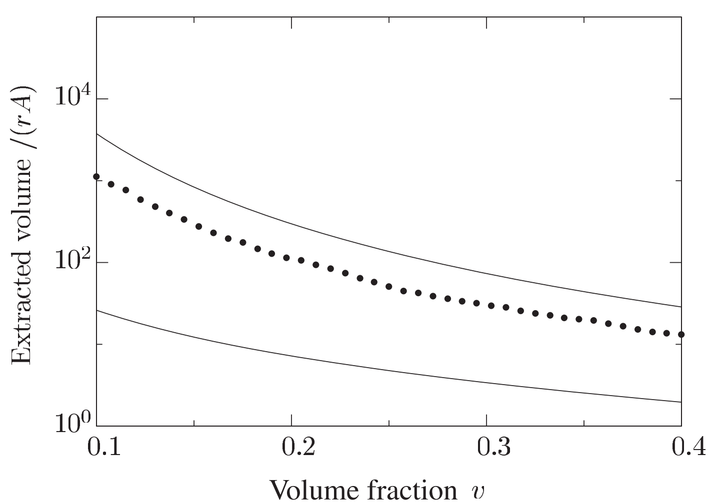

FIG. 6.1. *Normalized expected extracted volume $E(V^{\star})/(rA)$ (dotted line), normalized upper bound $E(U)/(rA)$ (upper solid line), and normalized lower bound $(E(L) - rA)/(rA)$ (lower solid line) versus volume fraction v (for $\lambda = 3$ and $\gamma = 1$).*

numerically computed with the method described in this paper. Note that, for a circular opening, $\lambda = 3$ when the radius of the orifice is twice the radius of the particles.

We have developed and analyzed a simple mathematical model to predict the volume of suspension extracted through a small orifice before it clogs. Our model leads to a simple and efficient numerical algorithm as well as analytic expressions for lower and upper bounds on the volume extracted. From the expressions of the bounds, our model reflects the sensitivity of the volume extracted to two key parameters: the volume fraction of particles, v, and λ, which reflects the ratio between the size of the orifice and the size of the particles.

A next step will be to validate the model (or relax some of the physical assumptions made) by comparing the predictions with experimental measurements. After the necessary adjustments, a more ambitious goal is to use the results obtained here as a building block to address more complex problems. These issues will be pursued in the future.

Appendix A. The expected value of the upper bound. To compute the upper bound on the expected volume of suspension extracted before clogging, we need the observations that we describe next. As in the rest of this paper, N is the number of particles initially placed in the container and $\mathcal{V}$ is the initial volume of the suspension.

Observation 1. *Let Ω be a region inside the container, and let $|\Omega|$ be its volume. The probability that the centers of exactly i of the N particles were initially placed in Ω is*

$$p(i, |\Omega|) = \frac{N!}{i!(N-i)!} \left(\frac{|\Omega|}{\mathcal{V}}\right)^i \left(1 - \frac{|\Omega|}{\mathcal{V}}\right)^{N-i}. \tag{A.1}$$

In particular, this probability depends only on i, N, and the volume of Ω. Moreover, the asymptotic value $p(i, |\Omega|)$ in the regime $N \gg i$ and $\mathcal{V} \gg |\Omega|$ is

$$p(i, |\Omega|) \simeq \frac{1}{i!} \left(\frac{N|\Omega|}{\mathcal{V}}\right)^i e^{-\frac{N|\Omega|}{\mathcal{V}}}. \tag{A.2}$$

In particular, if $|\Omega| = rA$,

$$p(i, rA) \simeq \frac{1}{i!} (\lambda v)^i e^{-\lambda v} \tag{A.3}$$

(where v is the volume fraction of the particles and λ was defined in (2.3)*).*

This observation results from the fact that the centers of the particles are placed in the container randomly with uniform probability distribution, from basic probability arguments (see any probability text book), and from the equality $rAN = \lambda v \mathcal{V}$.

We use the standard notation $P(z)$ for the probability that the event z is true.

Observation 2. *If i is an integer that satisfies $i \ll N$, and if $rA \ll \mathcal{V}$, then for any $0 \le V \le \mathcal{V} - rA$ we have*

$$P(k(V) = i) \simeq \frac{1}{i!} (\lambda v)^i e^{-\lambda v}. \tag{A.4}$$

Note, in particular, that $P(k(V) = i)$ is independent of V.

This observation results from the definition of the function $k = k(V)$ (see (2.1)), the fact that the volume of the set $\{x : V < F(x) \le V + rA\}$ is rA, and equation (A.3).

Observation 3. *Let Ω_1 and Ω_2 be two disjoint regions inside the container. Assume that $\mathcal{V} \gg \max\{|\Omega_1|, |\Omega_2|\}$. Let i_1 and i_2 be two nonnegative integers that satisfy $N \gg \max\{i_1, i_2\}$. The probability of having placed exactly i_1 centers of particles in Ω_1 and i_2 centers of particles in Ω_2 is asymptotically equal to $p(i_1, |\Omega_1|)p(i_2, |\Omega_2|)$.*

The validity of this observation is a consequence of the fact that the placements of exactly i_1 centers of particles in Ω_1 and i_2 centers of particles in Ω_2 are asymptotically independent events in the regime $N \gg \max\{i_1, i_2\}$ and $\mathcal{V} \gg \max\{|\Omega_1|, |\Omega_2|\}$.

Observation 4. *Let i and j be two different nonnegative integers, $i \neq j$. The random variables $k(irA)$ and $k(jrA)$ are asymptotically independent.*

This observation results from the definition of the function $k = k(V)$, the fact that the sets $\{x : jrA < F(x) \le (j+1)rA\}$ and $\{x : irA < F(x) \le (i+1)rA\}$ are disjoint, and Observation 3.

Let m be a nonnegative integer. From the definition of U (see (4.1)), we have $U = mrA$ if $k(jrA) \le k_{\max}$ for $0 \le j < m$ and $k(mrA) > k_{\max}$. Thus,

$$P(U = mrA) = P(k(jrA) \le k_{\max} \text{ for } j < m \text{ and } k(mrA) > k_{\max}). \tag{A.5}$$

Given Observation 4, the $m+1$ events $k(jrA) \le k_{\max}$ (for $0 \le j < m$) and $k(mrA) > k_{\max}$ are asymptotically independent (more precisely, in the parameter regime $mk_{\max} \ll N$). Thus, (A.5) reduces to

$$P(U = mrA) \simeq P(k(mrA) > k_{\max}) \prod_{j=0}^{m-1} P(k(jrA) \le k_{\max}). \tag{A.6}$$

From Observation 2 and the definition of the parameter μ in (4.4), we have that

$$P(k(jrA) \le k_{\max}) \simeq \mu \quad \text{and} \quad P(k(mrA) > k_{\max}) \simeq 1 - \mu. \tag{A.7}$$

Equations (A.6) and (A.7) imply that

$$P(U = mrA) \simeq (1-\mu)\mu^m, \tag{A.8}$$

and thus, in the parameter regime $N \gg k_{\max}$, the expected value of U is

$$E(U) \simeq \sum_{m=0}^{\infty} mrAP(U = mrA) \simeq rA \sum_{m=0}^{\infty} m(1-\mu)\mu^m = \frac{\mu}{1-\mu} rA, \tag{A.9}$$

which shows the validity of (4.4).

Appendix B. The upper bound in the regime $\lambda v \ll 1$. Given the definition of μ (see (4.4)), we have

$$e^{\lambda v}(1-\mu) = e^{\lambda v} - \sum_{i=0}^{k_{\max}} \frac{(\lambda v)^i}{i!} = \sum_{i=0}^{\infty} \frac{(\lambda v)^i}{i!} - \sum_{i=0}^{k_{\max}} \frac{(\lambda v)^i}{i!} = \sum_{i=[k_{\max}]+1}^{\infty} \frac{(\lambda v)^i}{i!}. \tag{B.1}$$

Thus, we have

$$1 - \mu = e^{-\lambda v} \frac{(\lambda v)^{[k_{\max}]+1}}{([k_{\max}]+1)!} \qquad \text{if} \quad \lambda v \ll 1. \tag{B.2}$$

Since we clearly have

$$\mu = e^{-\lambda v} \quad \text{if} \quad \lambda v \ll 1, \tag{B.3}$$

the validity of (4.5) follows.

Acknowledgments. The author thanks Professor Santamarina for introducing the author to this area of research and for stimulating discussions. This work was motivated by experiments of Santamarina's research group.

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