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BOUND ON THE YIELD SET OF FIBER REINFORCED COMPOSITES SUBJECTED TO ANTIPLANE SHEAR

Guillermo H. Goldsztein

School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160

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Abstract. We consider fiber reinforced composites where both the matrix and the fibers are made of ideally plastic materials with the fibers being much stronger than the matrix. We restrict our attention to microstructures and applied stresses that lead to both microscopic and macroscopic antiplane shear deformations. We obtain a bound on the yield set of the composite in terms of the shapes of the fibers, their volume fraction and the yield set of the matrix.

1. Introduction.

The stresses that an ideally plastic material can withstand form a bounded closed set $\bar{\mathbf{Y}}$ in the space of symmetric 3×3 real matrices. The set $\bar{\mathbf{Y}}$, which is a material property, is called the yield set or strength domain. Unlike brittle materials, ideally plastic materials do not break. When subjected to a stress that is in the boundary of $\bar{\mathbf{Y}}$, the material experiences a permanent deformation, usually called plastic deformation.

Fiber reinforced composites are materials made of solid fibers embedded in a weaker solid referred to as the matrix. We consider in this paper fiber reinforced composites where all the fibers are made of the same perfectly plastic homogeneous material. We denote by $\bar{\mathbf{Y}}_{\rm f}$ the yield set of the material the fibers are made of. We also assume that the matrix is made of a perfectly plastic solid with a different yield set $\bar{\mathbf{Y}}_{\rm m}$. Thus, the stresses within the fibers are restricted to the set $\bar{\mathbf{Y}}_{\rm f}$ but within the matrix the stresses are restricted to the set $\bar{\mathbf{Y}}_{\rm m}$. More precisely, denoting by Ω the region in space occupied by the composite, for each $\mathbf{x} \in \Omega$, we define

$$\bar{\mathbf{Y}}_{\mathbf{x}} = \begin{cases} \bar{\mathbf{Y}}_{\rm f} & \text{if } \mathbf{x} \text{ is in a fiber} \\ \bar{\mathbf{Y}}_{\rm m} & \text{if } \mathbf{x} \text{ is in the matrix.} \end{cases} \qquad (1)$$

If the composite is subject to a stress $\bar{\sigma}$ that may vary within the material $\bar{\sigma} = \bar{\sigma}(\mathbf{x})$, we have

$$\bar{\sigma}(\mathbf{x}) \in \bar{\mathbf{Y}}_{\mathbf{x}}. \qquad (2)$$

We will only consider time independent stresses $\bar{\sigma}$ and thus, $\bar{\sigma}$ also satisfies the equilibrium equations

$$\nabla \cdot \bar{\sigma} = 0, \qquad (3)$$

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where $\nabla \cdot \bar{\sigma}$ is the divergence of $\bar{\sigma}$.

Loosely speaking, the microstructure or microgeometry of the composite refers to the description of the regions in space occupied by the fibers and the matrix. More precisely, the microstructure is determined by the function $\mathbf{x} \mapsto \bar{\mathbf{Y}}_{\mathbf{x}}$.

Assume the microstructure $\bar{\mathbf{Y}}_{\mathbf{x}}$ is periodic with period cell $\bar{Q}$, where $\bar{Q}$ is a parallelepiped. A stress field $\bar{\sigma}$ is said to be admissible if it is $\bar{Q}$-periodic and satisfies the restrictions (2) and the equilibrium equations (3). In the limit in which the size of the period cell $\bar{Q}$ is much smaller than the size of the material Ω, the composite behaves *macroscopically* as a homogeneous material that can withstand only the stresses that belong to the set $\bar{\mathbf{Y}}_{\text{hom}}$ defined by

$$\bar{\mathbf{Y}}_{\text{hom}} = \{\bar{\tau} : \bar{\tau} = \langle \bar{\sigma} \rangle, \text{ for some } \bar{\sigma} \text{ admissible}\}, \tag{4}$$

where $\langle \bar{\sigma} \rangle$ is the average of $\bar{\sigma}$, i.e. $\langle \bar{\sigma} \rangle = |\bar{Q}|^{-1} \int_{\bar{Q}} \bar{\sigma}(\mathbf{x})\, \mathrm{d}\mathbf{x}$ with $|\bar{Q}|$ being the volume of $\bar{Q}$. In other words, $\bar{\mathbf{Y}}_{\text{hom}}$ is the set of *macroscopic* stresses that the composite can withstand. We will refer to $\bar{\mathbf{Y}}_{\text{hom}}$ as the yield set of the composite. The justification of equation (4) is a well understood fact of the theory of homogenization. We refer the reader to [5, 10, 11, 30, 52, 57, 58, 59] for more details.

In practice, all the details of the microstructure $\bar{\mathbf{Y}}_{\mathbf{x}}$ are not known or can not be controlled. Instead, only some partial information, such as the volume fraction of the fibers, is available. Accordingly, the general objective in the mathematical study of these type of materials has been to obtain bounds or estimates on $\bar{\mathbf{Y}}_{\text{hom}}$, the set of stresses the composite can withstand, in terms of the yield sets of the matrix and the fibers, and information that may be available about the microstructure. This is also our goal in this paper.

Fiber reinforced composites, where both the fibers and the matrix are (to a good approximation) ideally plastic, are very important and widely used in applications. Thus, their study, both theoretical and experimental, is a very active field of research.

From the mathematical point of view, fiber reinforced composites belong to the wider class of materials known as heterogeneous solids, which include composites and polycrystals. The same mathematical tools are sometimes used to study different heterogeneous solids. Ideally plastic materials are highly non-linear. Nevertheless, heterogeneous materials made of ideally plastic pure phases (the fibers and the matrix in our case) have proved to be amenable to detailed mathematical analysis. Thus, their study is very appealing since it can serve to test the existing mathematical methods, to develop new ones, and to gain intuition on the behavior of real materials. For the reasons mentioned in this and the previous paragraphs, the literature on the mathematical study of heterogeneous materials made of ideally plastic pure phases is very extensive. Some examples include [2, 3, 4, 14, 15, 16, 17, 18, 21, 22, 23, 24, 25, 27, 31, 32, 42, 43, 44, 45, 47, 48, 49, 53, 60, 65]. Examples of related non-linear homogenization problems include [20, 62, 66].

In our work, we will study the effect of the fiber shape on the yield set of the composite in the scalar two-dimensional problem that results from restricting our attention to applied stresses and microstructures that lead to both *microscopic* and *macroscopic* antiplane shear. More precisely, we compute a bound on the weakest direction of the yield set of this class of composites.

We mention that, while our results are completely new, the yield set of fiber or particle reinforced composites has been studied [1, 6, 7, 8, 9, 12, 13, 26, 28, 29, 33, 34, 35, 36, 37, 38, 46, 50, 51, 54, 55, 56, 61].

Our bound can be considered to be an application of a mathematical technique known as the translation method [41, 63, 64]. Other examples where the translation method was used to study non-linear composites or polycrystals include [19, 32, 21, 39, 40]. Particularly, the work in [32] has motivated a lot of work in this area of research.

This paper is organized as follows: In section 2 we reduce the original three-dimensional vectorial problem to a two-dimensional scalar problem by considering applied stresses and microstructures that lead to both microscopic and macroscopic antiplane shear. In section 3 we obtain a bound on the *strength* of the weakest direction of the composites we consider. We conclude in section 4 with some examples and a small discussion.

2. The microstructures, the yield set of the fibers and the matrix and antiplane shear.

2.1. The microstructures and the yield sets of the pure phases.

As usual, we denote by x_1, x_2 and x_3 the components of the position vector $\mathbf{x}$, i.e. $\mathbf{x} = (x_1, x_2, x_3)$. We assume that the fibers are infinite in length, they are parallel to the x_3-axis and the cross sections of the fibers (intersection of the fibers with a plane of the form $x_3 =$ constant) are independent of x_3. In other words, the function $\mathbf{x} \mapsto \bar{\mathbf{Y}}_{\mathbf{x}}$ is independent of x_3. Note that, as a consequence, the period cell $\bar{Q}$ is of the form $\bar{Q} = Q \times [a, b]$, where Q is a parallelogram in $\mathbb{R}^2$ and a and b are any numbers that satisfy $a < b$.

We also assume that the yield sets $\bar{\mathbf{Y}}_{\mathrm{f}}$ and $\bar{\mathbf{Y}}_{\mathrm{m}}$ are convex sets with the following symmetric property: For both $\bar{\mathbf{Y}} = \bar{\mathbf{Y}}_{\mathrm{f}}$ and $\bar{\mathbf{Y}} = \bar{\mathbf{Y}}_{\mathrm{m}}$,

$$\text{if } \begin{bmatrix} \bar{\sigma}_{11} & \bar{\sigma}_{12} & \bar{\sigma}_{13} \\ \bar{\sigma}_{21} & \bar{\sigma}_{22} & \bar{\sigma}_{23} \\ \bar{\sigma}_{31} & \bar{\sigma}_{32} & \bar{\sigma}_{33} \end{bmatrix} \text{ belongs to } \bar{\mathbf{Y}}, \text{ so does } \begin{bmatrix} -\bar{\sigma}_{11} & -\bar{\sigma}_{12} & \bar{\sigma}_{13} \\ -\bar{\sigma}_{21} & -\bar{\sigma}_{22} & \bar{\sigma}_{23} \\ \bar{\sigma}_{31} & \bar{\sigma}_{32} & -\bar{\sigma}_{33} \end{bmatrix}. \tag{5}$$

This assumption is not restrictive at all. We refer the reader to [32] and references therein for a discussion of this symmetry condition.

2.2. Antiplane shear.

Due to the assumptions of subsection 2.1, the two-dimensional scalar problem that corresponds to antiplane shear is well defined. More precisely, for $i = \mathrm{f}$, $i = \mathrm{m}$ and $i = \mathrm{hom}$, we define the sets

$$\mathbf{Y}_i = \left\{ \sigma = (\sigma_1, \sigma_2) \in \mathbb{R}^2 : \begin{bmatrix} 0 & 0 & \sigma_1 \\ 0 & 0 & \sigma_2 \\ \sigma_1 & \sigma_2 & 0 \end{bmatrix} \in \bar{\mathbf{Y}}_i \right\}. \tag{6}$$

For each $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, we also define

$$\mathbf{Y}_{\mathbf{x}} = \begin{cases} \mathbf{Y}_{\mathrm{f}} & \text{if } \{\mathbf{x}\} \times \mathbb{R} \text{ is included in a fiber} \\ \mathbf{Y}_{\mathrm{m}} & \text{if } \{\mathbf{x}\} \times \mathbb{R} \text{ is included in the matrix.} \end{cases} \tag{7}$$

We say that a two-dimensional vector field $\sigma = \sigma(\mathbf{x}) = (\sigma(x_1, x_2), \sigma(x_1, x_2))$ is admissible if it is Q-periodic and it satisfies the restrictions

$$\sigma(\mathbf{x}) \in \mathbf{Y}_{\mathbf{x}} \text{ for all } \mathbf{x} \in \mathbb{R}^2 \tag{8}$$

and the equilibrium equations

$$\nabla \cdot \sigma = 0, \tag{9}$$

where $\nabla \cdot \sigma$ is the divergence of σ in two dimensions.

It can be easily shown that, given the conditions stated in section 2.1

$$\mathbf{Y}_{\text{hom}} = \{\tau : \tau = \langle\sigma\rangle, \text{ for some two-dimensional admissible vector field } \sigma\}, \quad (10)$$

where $\langle\sigma\rangle$ is now the two-dimensional average of σ, i.e. $\langle\sigma\rangle = |Q|^{-1}\int_Q \sigma(\mathbf{x})\,\mathrm{d}\mathbf{x}$ with $|Q|$ being the area of Q.

All the vector fields we will consider in the rest of this paper are two-dimensional. The two-dimensional sets $\mathbf{Y}_{\text{f}}$, $\mathbf{Y}_{\text{m}}$ and $\mathbf{Y}_{\text{hom}}$ will also be referred as the yield sets of the fibers, the matrix and the composite respectively.

2.3. **The yield sets of the pure phases.** We will assume that the sets $\mathbf{Y}_{\text{f}}$ and $\mathbf{Y}_{\text{m}}$ are circles centered at the origin with radius Y_{f} and Y_{m} respectively. Thus, defining

$$Y_{\mathbf{x}} = \begin{cases} Y_{\text{f}} & \text{if } \{\mathbf{x}\}\times\mathbb{R} \text{ is included in a fiber} \\ Y_{\text{m}} & \text{if } \{\mathbf{x}\}\times\mathbb{R} \text{ is included in the matrix.} \end{cases} \quad (11)$$

we have that a vector field σ is admissible if and only if σ is Q-periodic, $\|\sigma(\mathbf{x})\| \le Y_{\mathbf{x}}$ for all $\mathbf{x}\in\mathbb{R}^2$ and $\nabla\cdot\sigma = 0$, where $\|.\|$ denotes the euclidean norm, i.e. $\|(z_1, z_2)\| = \sqrt{z_1^2 + z_2^2}$ for all $\mathbf{z} = (z_1, z_2) \in \mathbb{R}^2$.

Note that divergence free vector fields can be regarded as the velocity field of an incompressible fluid. Thus, in the rest of this paper, we will sometimes refer to two-dimensional divergence free vector fields as stresses and sometimes as fluid velocity fields.

3. **Bound on the yield set of the composites.** Given any vector $\mathbf{z} = (z_1, z_2)$ we denote by $\mathbf{z}^\perp$ the vector perpendicular to $\mathbf{z}$ that results from rotating $\mathbf{z}$ an angle of $\pi/2$ in the counterclockwise direction $\mathbf{z}^\perp = (-z_2, z_1)$. Also, $\cdot$ denotes the dot product, i.e. $\mathbf{u}\cdot\mathbf{w} = u_1w_1 + u_2w_2$ for all two-dimensional vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{w} = (w_1, w_2)$.

Observation 3.1. *Let σ and α be two Q-periodic divergence-free vector fields. Then,*

$$\langle\sigma\cdot\alpha^\perp\rangle = \langle\sigma\rangle\cdot\langle\alpha\rangle^\perp. \quad (12)$$

Since this is a well known fact, we do not prove it here (see [32] for details).

We will refer to the cross sections of the fibers as the inclusions. In other words, the inclusions are the connected components of the set $\{\mathbf{x}\in\mathbb{R}^2 : Y_{\mathbf{x}} = Y_{\text{f}}\}$.

Assumptions 3.1. *For any inclusion I, we will assume in the rest of this paper that the number of points that belong to the boundary of I and also to the boundary of an other inclusion is finite.*

Observation 3.2. *Let I be an inclusion and P_I its perimeter, i.e. P_I is the length of the boundary of I. If σ and α are two admissible vector fields then,*

$$\left|\int_I \sigma(\mathbf{x})\cdot\alpha^\perp(\mathbf{x})\,\mathrm{d}\mathbf{x}\right| \le \frac{1}{2}\left(Y_{\text{m}}P_I\right)^2. \quad (13)$$

Proof. We first note that, since σ and α are divergence-free, there exist $\psi = \psi(\mathbf{x})$ and $\phi = \phi(\mathbf{x})$, two scalar functions defined in $\mathbb{R}^2$, such that

$$\sigma = (\nabla\psi)^\perp \text{ and } \alpha = (\nabla\phi)^\perp, \quad (14)$$

where $\nabla\psi$ and $\nabla\phi$ are the gradients of ψ and ϕ respectively. Using this last equation, the divergence theorem, the fact that the dot product of two vectors does not change if we rotate the two vectors by the same angle, and the fact that $(\mathbf{z}^\perp)^\perp = -\mathbf{z}$ for all vectors $\mathbf{z}$, we obtain

$$\int_I \sigma \cdot \alpha^\perp = -\int_I (\nabla\psi)^\perp \cdot \nabla\phi = -\int_{\partial I} \phi\,(\nabla\psi)^\perp \cdot \hat{n} = \int_{\partial I} \phi\nabla\psi \cdot \hat{n}^\perp, \tag{15}$$

where ∂I is the boundary of I and $\hat{n} = \hat{n}(\mathbf{x})$ is the vector of norm one perpendicular to ∂I at $\mathbf{x}$ that points outward I.

Let $\mathbf{x} = \mathbf{x}(s)$ be a parametrization of ∂I in the counterclockwise direction. Let $\mathbf{x}'(s)$ is the derivative of $\mathbf{x}(s)$ with respect to s. We assume that ∂I is regular enough so that the parametrization $\mathbf{x}(s)$ can be chosen to satisfy $\|\mathbf{x}'(s)\| = 1$ for all $0 \leq s \leq P_I$ except probably in a finite number of points s where $\mathbf{x}'(s)$ is not defined. Thus, it is easy to verify that the following is true:

$$\hat{n}^\perp(\mathbf{x}(s)) = \mathbf{x}'(s) \tag{16}$$

and

$$\phi(\mathbf{x}(s)) = \phi(\mathbf{x}(0)) + \int_0^s \nabla\phi(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\mathrm{d}t. \tag{17}$$

Using the last three equations, we obtain

$$\int_I \sigma \cdot \alpha^\perp = \int_0^{P_I} \left(\phi(\mathbf{x}(0)) + \int_0^s \nabla\phi(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\mathrm{d}t\right) \nabla\psi(\mathbf{x}(s)) \cdot \mathbf{x}'(s)\mathrm{d}s =$$
$$= \int_0^{P_I} \nabla\psi(\mathbf{x}(s)) \cdot \mathbf{x}'(s) \left(\int_0^s \nabla\phi(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\mathrm{d}t\right) \mathrm{d}s. \tag{18}$$

Note that the assumption 3.1 implies that, for all $s \in [0, P_I]$ except probably a finite set, $\mathbf{x}(s)$ is in the boundary of the interior of the matrix. Using this fact, equations (14) and (16), the fact that σ is admissible, and the fact that the value of $\sigma \cdot \hat{n}$ at the inclusion boundary is the same in the inclusion side as in the matrix side, we get

$$|\nabla\psi(\mathbf{x}(s)) \cdot \mathbf{x}'(s)| = |\sigma(\mathbf{x}(s)) \cdot \hat{n}(\mathbf{x}(s))| \leq \|\sigma(\mathbf{x}(s))\| \leq Y_{\mathrm{m}}. \tag{19}$$

Similarly, $|\nabla\phi(\mathbf{x}(t)) \cdot \mathbf{x}'(t)| \leq Y_{\mathrm{m}}$. Thus, going back to equation (18) and with simple calculations we obtain the validity of equation (13). □

The period cell Q is a parallelogram and thus, there are two vectors $\mathbf{u}, \mathbf{w} \in \mathbb{R}^2$ such that

$$Q = \left\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = s\mathbf{u} + t\mathbf{w} \text{ for some } 0 \leq s < 1 \text{ and } 0 \leq t < 1\right\}. \tag{20}$$

We say that two inclusions I_1 and I_2 are equivalent if there exists integers k and ℓ such that $I_2 = I_1 + k\mathbf{u} + \ell\mathbf{w}$. We denote by $\mathbb{I}$ a set of inclusions that contains exactly one inclusion per equivalent class. We will need the set $\mathbb{I}$ when we take averages. For example, one choice of $\mathbb{I}$ is the set of inclusions whose center of mass is in Q.

Observation 3.3. *Let ν be the volume fraction of the inclusions, i.e.*

$$\nu = \frac{|\{\mathbf{x} \in Q \text{ and } \mathbf{x} \in \text{an inclusion}\}|}{|Q|}. \tag{21}$$

We define the parameter

$$\eta = \max_{I \text{ inclusion}} \frac{P_I^2}{|I|}, \tag{22}$$

where as before, P_I is the length of the boundary of the inclusion I. If σ and α are two admissible vector fields, then

$$\left|\langle\sigma\rangle \cdot \langle\alpha\rangle^{\perp}\right| \leq (1-\nu)\, Y_{\mathrm{m}}^2 + \frac{1}{2}\nu\eta Y_{\mathrm{m}}^2. \tag{23}$$

Proof. From observations 3.1 and 3.2 it can be easily shown that

$$\left|\langle\sigma\rangle \cdot \langle\alpha\rangle^{\perp}\right| \leq (1-\nu)\, Y_{\mathrm{m}}^2 + \frac{Y_{\mathrm{m}}^2}{2|Q|} \sum_{I \in \mathbb{I}} P_I^2. \tag{24}$$

We use this last equation and the fact that, for all inclusions I, $P_I^2 \leq \eta |I|$ to get

$$\left|\langle\sigma\rangle \cdot \langle\alpha\rangle^{\perp}\right| \leq (1-\nu)\, Y_{\mathrm{m}}^2 + \frac{Y_{\mathrm{m}}^2}{2|Q|} \eta \sum_{I \in \mathbb{I}} |I|. \tag{25}$$

Equation (23) follows once we note that $\nu = |Q|^{-1} \sum_{I \in \mathbb{I}} |I|$. □

Observation 3.4. *Let τ be a vector such that both τ and $\tau^{\perp}$ belong to $\mathbf{Y}_{\mathrm{hom}}$. Then,*

$$\|\tau\| \leq Y_{\mathrm{m}} \sqrt{(1-\nu) + \frac{1}{2}\nu\eta}, \tag{26}$$

where ν is the volume fraction of the inclusions and η is as defined in equation (22).

Proof. Since both τ and $\tau^{\perp}$ belong to $\mathbf{Y}_{\mathrm{hom}}$, there exist σ and α admissible vector fields such that $\tau = \langle\sigma\rangle$ and $\tau^{\perp} = \langle\alpha\rangle$. Applying equation (23) and taking square root completes the proof of this observation. □

When reinforcing a material, it is usually desirable that the material is strong in all directions. Thus, it is of interest to evaluate *strength* of the weakest direction of the composite. This motivates the following definition.

Definition 3.1. Let $\mathbf{e} \in \mathbb{R}^2$ be a vector of length 1. We say that the strength of the composite in the direction $\mathbf{e}$ is $S_{\mathbf{e}}$ if $t\,\mathbf{e} \in \mathbf{Y}_{\mathrm{hom}}$ for all $t < S_{\mathbf{e}}$ and $t\,\mathbf{e} \notin \mathbf{Y}_{\mathrm{hom}}$ for all $t > S_{\mathbf{e}}$.

Note that such $S_{\mathbf{e}}$ is well defined because $\mathbf{Y}_{\mathrm{hom}}$ in convex.

Our bound will in fact be a bound on the strength of the weakest direction of the composite, which we define next.

Definition 3.2. The strength of the weakest direction of the composite is

$$S_{\mathrm{hom}}^{\mathrm{weak}} = \inf_{\{\mathbf{e} \in \mathbb{R}^2 : \|\mathbf{e}\| = 1\}} S_{\mathbf{e}}. \tag{27}$$

A little thought will convince the reader that $S_{\mathrm{hom}}^{\mathrm{weak}}$ is the strength of the weakest direction of the composite. The composite can withstand stresses of norm less than $S_{\mathrm{hom}}^{\mathrm{weak}}$ in all directions. On the other hand, if $t > S_{\mathrm{hom}}^{\mathrm{weak}}$, there are stresses of norm t that the composite can not withstand.

The main result of this paper, which is a direct consequence of Observation 3.4, is now stated in the following theorem.

Theorem 3.3. *The strength of the weakest direction of the composite, $S_{\mathrm{hom}}^{\mathrm{weak}}$, satisfies the following bound:*

$$S_{\mathrm{hom}}^{\mathrm{weak}} \leq Y_{\mathrm{m}} \sqrt{(1-\nu) + \frac{1}{2}\nu\eta}, \tag{28}$$

where ν is the volume fraction of the inclusions and η is as defined in equation (22).

4. **Examples and conclusions.**

4.1. **Circular inclusions.** The square of the perimeter of a circle divided by its area is equal to 4π. Thus,

$$\eta = 4\pi \qquad \text{if all the inclusions are circular,} \tag{29}$$

where η is the parameter defined in equation (22). As a consequence, our bound (equation (28)) becomes

$$S^{\text{weak}}_{\text{hom}} \leq Y_{\text{m}}\sqrt{(1-\nu)+2\nu\pi} \qquad \text{if all the inclusions are circular,} \tag{30}$$

where, as before, ν is the volume fraction of the inclusions.

Note that, since $0 < \nu < 1$, Equation (30) implies $S^{\text{weak}}_{\text{hom}} \leq Y_{\text{m}}\sqrt{2\pi}$. In other words, the strength of the weakest direction of a composite with circular inclusions is never greater that $\sqrt{2\pi}$ times the Y_{m}, the strength of the matrix.

4.2. **Slender inclusions.** The parameter η becomes large when the inclusions are slender. We will provide an example now. A natural choice would be to consider elliptical inclusions. However, to make the calculations more explicit, we will assume the inclusions to be rectangular. For each inclusion I, let γ_I be the length of one of the longest sides of I divided by the length of one of shortest sides of I. Assume that γ_I is the same value for all inclusions I. Denote this value by γ, i.e. $\gamma = \gamma_I$ for any inclusion I. Thus, for each inclusions, the square of its perimeter divided by its area is equal to $4(\gamma+1)^2/\gamma$. To satisfy the assumption 3.1, we assume that the boundary of two different inclusions intersect at most in one point. We are particularly interested in the limit $\gamma \gg 1$. Thus,

$$\eta = 4\frac{(\gamma+1)^2}{\gamma} \approx 4\gamma \text{ for } \gamma \gg 1, \tag{31}$$

where η is the parameter defined in equation (22). As a consequence, our bound (equation (28)) becomes

$$S^{\text{weak}}_{\text{hom}} \leq Y_{\text{m}}\sqrt{(1-\nu)+2\nu\frac{(\gamma+1)^2}{\gamma}} \approx Y_{\text{m}}\sqrt{(1-\nu)+2\nu\gamma} \text{ for } \gamma \gg 1, \tag{32}$$

where, as before, ν is the volume fraction of the inclusions.

4.3. **Sharpness of the bound.** In this subsection, we continue our discussion of subsection 4.2, i.e. the inclusions are rectangular and γ_I and γ are as defined in subsection 4.2.

For this class of composites (the inclusions are rectangular), our bound increases with γ as $\sqrt{\gamma}$. The natural question that arises is: *Does the strength of the weakest direction in fact increases as* $\sqrt{\gamma}$. In other words: *Is our bound sharp?*. The anwser to these questions requires a careful study that will be presented by the author in the future elsewhere. However, we can preview here the anwser: *Yes, our bound is sharp in the sense that for each* γ *there exists a composite with rectangular inclusions such that,* $\gamma_I = \gamma$ *for all its inclusions and the strength of the weakest direction is larger that* $C\sqrt{\gamma}$*, where* C *is a constant independent of* γ.

4.4. **Final remarks.** In the two-dimensional scalar context of this paper, we were able to obtain a bound on the weakest direction of matrix-inclusion composites when the matrix and the inclusions are ideally plastic. This bound is a simple explicit formula (equation (22)). This formula depends only on $Y_{\rm m}$, the yield stress of matrix, ν, the volume fraction of the inclusions and η, a geometric parameter that depends on the shape of the inclusions. Note that our bound is independent of $Y_{\rm f}$, the yield stress of the inclusions. Thus, the strength of the composite reaches a plateau as the strength of the inclusions, $Y_{\rm f}$, increases.

In the broader context of techniques to compute effective properties of composites, our work is an example where the translation method was successful to provide a bound in the form of a simple formula. Our work is also an example that shows that ideally plastic heterogeneous materials are amenable to analysis leading to new insight. In the particular case of this work, we have identified the *microscopic* parameter η, which seems to have an effect on the *macroscopic* behavior of the composite.

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E-mail address: ggold@math.gatech.edu