110.302 February 18, 2000
Differential Equations
Summer 1994, JHU

FINAL EXAM

Time: 180min

Find the general solution of the following differential equations:

1. $xdy + (3 - 2x^2 - y)dx = 0.$

2. $(x - y)dx - (x + y)dy = 0.$

3. $xy'' + y' = 4x.$

4. $x^2y'' + 2xy' - 2y = 0.$

5. $y'' + 9y = 4sinx - 26e^{-2x} + 27x^3.$

6. $y^{(6)} - 2y^3 + 1 = 11x^2 + 22x + 33.$

7. Find the general solution of :

$$xy'' + 2y' + xy = 0.$$

Justify your answer and if possible, express it in terms of elementary functions.

8. The *Chebyshev's Equation* is:

$$(1 - x^2)y'' - xy' + p^2y = 1$$

For p=1, find two linerly independent solutions valid for $|x| < 1$.Justify your answer.

9. Solve the following linear system:

$$\frac{dx}{dt} = -4x - y$$

$$\frac{dy}{dt} = x - 2y \ .$$

10. Show that the condition $a_2 b_1 > 0$ is *sufficient*, but not *necessary* for the system:

$$\frac{dx}{dt} = a_1 x + b_1 y$$
$$\frac{dy}{dt} = a_2 x + b_2 y$$

to have two linearly independent real solutions.

11. A tank contains 10 gallons of *brine* in which 2 pounds of salt are dissolved. Brine containing 1 pound of salt per gallon is pumped into the tank at the rate of 3 gallons/minute, and the stirred mixture is drained off at the rate of 4 gallons/minute. Find the amount of salt in the tank at time t.

12. Let $y' = f(x, y)$ be a *homogeneous differential equation* and prove the following geometric fact about its family of integral curves. If the xy-plane is stretched from the origin by an scale of k, i.e. (x, y) is sent to $(x_1, y_1) = (kx, ky)$, then every integral curve C is moved to an integral curve C_1. Give an example.

14. A *cylindrical buoy* floats with its axis vertical in water. If it is depressed slightly, a restoring force equal to the weight of the displaced water presses it upward; and if it is then released, it will bop up and down. Find the period of oscillation if the friction of water is neglected.

15. *Voltera's prey-predator equations* are

$$\frac{dx}{dt} = x(a - by)$$

$$\frac{dy}{dt} = y(dx - c)$$

Find the equilibrium point for this dynamical system and discuss its significance. Next, linearize the system and find the orbits.