SOLUTIONS FOR MIDTERM 1

(1) (a)

$$-1 \leq \cos x \leq 1$$

$$\frac{-1}{x} \leq \frac{\cos x}{x} \leq \frac{1}{x}$$

$$\lim_{x\to\infty} \frac{-1}{x} \leq \lim_{x\to\infty} \frac{\cos x}{x} \leq \lim_{x\to\infty} \frac{1}{x}$$

$$(By\ the\ Squeeze\ Theorem)$$

$$0 \leq \lim_{x\to\infty} \frac{\cos x}{x} \leq 0$$

$$\lim_{x\to\infty} \frac{\cos x}{x} = 0$$

(b) If $x \leq 0$, then $|x| = -x$.

$$\lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} \frac{x}{-x} = lim_{x\to 0^-} - 1 = -1$$

(c)

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)} = \lim_{x\to 2}(x+2) = 4$$

(d)

$$\lim_{x\to\infty} \frac{\sqrt{2x+1}}{x+4} = \lim_{x\to\infty} \frac{\frac{\sqrt{2x+1}}{x}}{\frac{x+4}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{2x}{x^2}+\frac{1}{x^2}}}{\frac{x}{x}+\frac{4}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{2}{x}+\frac{1}{x^2}}}{1+\frac{4}{x}} = \frac{\sqrt{0+0}}{1+0} = 0$$

(2)

$$f(x) = \frac{\cos x}{\sin x}$$

$$f'(x) = \frac{\cos' x \sin x - \cos x \sin' x}{\sin^2 x}$$

$$(by\ the\ Quotient\ Rule)$$

$$= \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x$$

(3) We need to show that for every $\epsilon > 0$, there is a $\delta > 0$ such that $0 < |x+4| < \delta \Longrightarrow |(3x-7)-5| < \epsilon$.
Preliminary Analysis:

$$\begin{aligned} |(3x-7)-5| &< \epsilon \\ |3x-12| &< \epsilon \\ 3|x-4| &< \epsilon \\ |x-4| &< \frac{\epsilon}{3}, \textit{ so set } \delta \leq \frac{\epsilon}{3}. \end{aligned}$$

Formal Proof:

$$\begin{aligned} |x-4| &< \delta \\ |x-4| &< \frac{\epsilon}{3} \\ 3|x-4| &< \epsilon \\ |3x-12| &< \epsilon \\ |(3x-7)-5| &< \epsilon \end{aligned}$$

So, if we need $\epsilon < 0.01$, we set $\delta \leq \frac{0.01}{3}$.

(4)

$$\begin{aligned} f(x) &= x^5 + 4x^3 - 7x + 14 \\ f(1) &= 1 + 4 - 7 + 14 = 12 > 0 \\ f(-2) &= -32 - 32 + 14 + 14 = -36 < 0 \end{aligned}$$

So, since f is continuous, there exists $1 \leq c \leq -2$ such that $f(c) = 0$, by the Intermediate Value Theorem (IVT).

(5)

$$\begin{aligned} f(x) &= \frac{1}{x} \\ f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h\to 0} \frac{\frac{x-x-h}{x(x+h)}}{\frac{h}{1}} \\ &= \lim_{h\to 0} \frac{-1}{x(x+h)} \\ &= \lim_{h\to 0} \frac{-1}{x^2+xh} = \frac{-1}{x^2} \end{aligned}$$

(6)

$$\begin{aligned} y &= x^2 - 2x + 2, && (1,1) \\ \frac{dy}{dx} &= 2x - 2 && \Longrightarrow \quad m = 2(1) - 2 = 0 \\ y - 1 &= m(x-1) && \Longrightarrow \quad y - 1 = 0(x-1) \\ & && \Longrightarrow \quad y - 1 = 0 \\ & && \Longrightarrow \quad y = 1 \end{aligned}$$

(7) Let x denote the position of the car, and $V(x)$ be the velocity of the car at x.

Set

$$f(x) = V(x) - x).$$

Then,

$$f(0) = V(0) - 0 = V(0) \geq 0$$

and

$$f(100) = V(100) - 100 \leq 0,$$

because, by assumption, $V(x) \leq 100$. So, since f is continuous, there exists $0 \leq c \leq 100$ such that $f(c) = 0$, by IVT. In particular, $V(c) = c$ for some $0 \leq c \leq 100$. □