Math 426 Jan 27, 2004[1]
Introduction to Modern Geometry
Spring 2004, PSU

Lecture Notes 2

1.3 Curvature

The curvature of a curve is a measure of how fast it is turning. More precisely, it is the speed, with respect to the arclength parameter, of the unit tangent vector of the curve. The unit tangent vector, a.k.a. *tangential indicatrix*, or *tantrix* for short, of a regular curve $\alpha\colon I \to \mathbf{R}^n$ is defined as

$$T(t) := \frac{\alpha'(t)}{\|\alpha'(t)\|}.$$

Note that the tantrix is itself a curve with parameter ranging in I and image lying on the unit sphere $\mathbf{S}^{n-1} := \{x \in \mathbf{R}^n \mid \|x\| = 1\}$. If α is parametrized with respect to arclength, i.e., $\|\alpha'(t)\| = 1$, then the curvature is given by

$$\kappa(t) = \|T'(t)\| = \|\alpha''(t)\| \quad \text{(provided } \|\alpha'\| = 1\text{)}.$$

Thus the curvature of a road, is the amount of centripetal force which you would feel, if you traveled on it in a car which has unit speed; the tighter the turn, the higher the curvature, as is affirmed by the following exercise:

Exercise 1. Show that the curvature of a circle of radius r is $\frac{1}{r}$, and the curvaure of the line is zero (First you need to find arclength parametrizations for these curves).

As we showed earlier, any curve may be reparametrized with respect to arclength, thus the above definition is not restrictive; however, as a practical matter, we need to have a definition for curvature which works for all curves, becuase it is often very difficult, or even impossible, to find explicit formulas for unit speed curves.

To find the general formula for curvature, let $\alpha\colon I \to \mathbf{R}^n$ be a general regular curve (not necessarily parametrized by arclength) and let $T\colon I \to$

[1]Last revised: January 29, 2004

$\mathbf{S}^{n-1}$ be its tantrix. Let $s\colon I \to [0, L]$ be the arclength function. Since, as we discussed earlier s is invertible, we may define

$$\overline{T} := T \circ s^{-1}$$

to a reparametrization of T. Then curvature may be defined as

$$\kappa(t) := \|\overline{T}'(s(t))\|.$$

By the chain rule,

$$\overline{T}'(t) = T'\Big(s^{-1}(t)\Big) \cdot \Big(s^{-1}\Big)'(t).$$

Further recall that $(s^{-1})'(t) = 1/\|\alpha'(s^{-1}(t))\|$. Thus

$$\kappa(t) = \frac{\|T'(t)\|}{\|\alpha'(t)\|}.$$

Exercise 2. Use the above formula, together with defintion of T, to show that

$$\kappa(t) = \frac{\sqrt{\|\alpha'(t)\|^2\|\alpha''(t)\|^2 - \langle \alpha'(t), \alpha''(t)\rangle^2}}{\|\alpha'(t)\|^3}.$$

In particular, in $\mathbf{R}^3$, we have

$$\kappa(t) = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3}.$$

(*Hint*: The first identity follows from a straight forward computation. The second identity is an immediate result of the first via the identity $\|v \times w\|^2 = \|v\|^2\|w\|^2 - \langle v, w\rangle^2$.)

Exercise 3. Show that the curvature of a planar curve which satisfies the equation $y = f(x)$ is given by

$$\kappa(x) = \frac{|f''(x)|}{\left(\sqrt{1 + \big(f'(x)\big)^2}\,\right)^3}.$$

(*Hint:* Use the parametrization $\alpha(t) = (t, f(t), 0)$, and use the formula in previous exercise.) Compute the curvatures of $y = x$, x^2, x^3, and x^4.