Math 550 Sep 12, 2001
Vector Analysis
Fall 2001, USC

PRACTICE QUIZ 2

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1. (Law of sines) Use cross products to prove the identity:

$$\sin(\theta + \phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi).$$

Hints: Let $u = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}$, and $v = \cos(\phi)\mathbf{i} - \sin(\phi)\mathbf{j}$. Sketch these vectors, for small θ and ϕ, to see that the angle between them is $\theta + \phi$, and compute the norm of $u \times v$.

2. (Pythagorean theorem in 3D) Let $abcd$ be a tetrahedron, and A, B, C, and D be the area of the faces opposite to the vertices a, b, c, and d respectively. Suppose that the three adjacent faces at the vertex a all have a right angle at a. Show that

$$A^2 = B^2 + C^2 + D^2.$$

Hints: Let $a = (0, 0, 0)$, and note that $A^2 = \frac{1}{2}\|\overrightarrow{bc} \times \overrightarrow{bd}\|^2$.

3. Let $abcd$ be an arbitrary tetrahedron and A, B, C, and D be as in the previous problem. Let u_a be a unit vector which is orthogonal to the face bcd, and points outside of the tetrahedron. Similarly, define u_b, u_c, and u_d. Show that

$$Au_a + Bu_b + Cu_c + Du_d = 0.$$

Hints: Let u, v, w denote 3 adjacent edges of the tetrahedron, and write each of the terms in the above equation as an appropriate cross product (recall the right hand rule to get the directions right).

4. Show that a result similar to the formula in problem 3 holds for all convex polytopes such as the cube or any other of the platonic solids. *Hint:* These solids are decomposable into tetrahedra.

Note: The converse of problem of 4 is also true. That is, given n unit vectors u_i, and numbers A_i such that $\sum_{i=1}^{n} A_i u_i = 0$, there exists a convex polytope with n faces which have area A_i and are perpendicular to u_i (this is a theorem of Minkowski).

5. [Extra Credit] For an arbitrary tetrahedron $abcd$, prove that

$$\|ac\|^2 + \|bd\|^2 \leq \|bc\|^2 + \|ad\|^2 + 2\|ab\|\|cd\|,$$

and equality holds if and only if $\vec{ab}$ and $\vec{dc}$ are parallel.