Math 550 Sep 17, 2001
Vector Analysis
Fall 2001, USC

Solutions to Midterm 1

1. No, becuase:

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = \mathbf{i} \times (-\mathbf{j}) = -\mathbf{k}, \qquad \text{but} \qquad (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = (0,0,0) \times \mathbf{k} = 0.$$

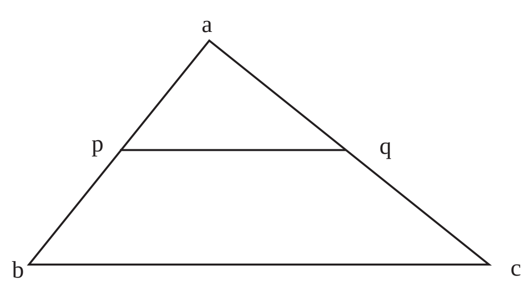
2.

$$\theta = \cos^{-1} \frac{(1,1,0) \cdot (1,1,2)}{\|(1,1,0)\| \, \|(1,1,\sqrt{2})\|} = \cos^{-1} \frac{1}{\sqrt{2}} = \frac{\pi}{4}.$$

3. Let $a = (1,1,1)$, $b = (1,1,2)$, and $c = (1,2,1)$. Then

$$A = \frac{1}{2}\|(b-a) \times (c-a)\| = \frac{1}{2}\|\mathbf{i} \times \mathbf{j}\| = \frac{1}{2}\|\mathbf{k}\| = \frac{1}{2}.$$

4.



$$\vec{pq} = q - p = \frac{a+b}{2} - \frac{a+c}{2} = \frac{b-c}{2} = \frac{1}{2}\vec{bc}$$

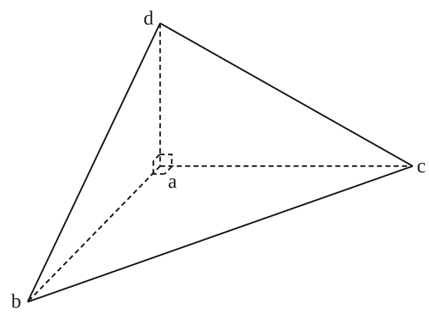
5. Let $p = (3,3,0)$, $p_0 = (2,3,5)$ and $\mathbf{n} = (1,1,1)$.

$$d = \|\operatorname{Proj}_{\mathbf{n}} \vec{pp_0}\| = \left| \frac{(p - p_0) \cdot \mathbf{n}}{\|\mathbf{n}\|} \right| = \left| \frac{(1,0,-5) \cdot (1,1,1)}{\sqrt{3}} \right| = \frac{4}{\sqrt{3}}$$

6. Let $\mathbf{a} = (a_1, a_2, a_3)$, and $\mathbf{b} = (b_1, b_2, b_3)$. By the Cauchy-Schwarts inequality, $(\mathbf{a} \cdot \mathbf{b})^2 \leq \|\mathbf{a}\|^2 \, \|\mathbf{b}\|^2$, which yields:

$$(a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2)\,(b_1^2 + b_2^2 + b_3^2).$$

7. Let $\mathbf{a} = (0, 0, 0)$, $\mathbf{b} = b\,\mathbf{i}$, $\mathbf{c} = c\,\mathbf{j}$, $\mathbf{d} = d\,\mathbf{k}$.



$$\begin{aligned} A^2 &= \frac{1}{4}\|\vec{bd} \times \vec{cd}\|^2 = \frac{1}{4}\|(d\,\mathbf{k} - b\,\mathbf{i}) \times (d\,\mathbf{k} - c\,\mathbf{j})\|^2 \\ &= \frac{1}{4}\|d^2\,\mathbf{k} \times \mathbf{k} - dc\,\mathbf{k} \times \mathbf{j} - bd\,\mathbf{i} \times \mathbf{k} + bc\,\mathbf{i} \times \mathbf{j}\|^2 \\ &= \frac{1}{4}\|dc\,\mathbf{i} + bd\mathbf{j} + bc\,\mathbf{k}\|^2 = \left(\frac{dc}{2}\right)^2 + \left(\frac{bd}{2}\right)^2 + \left(\frac{bc}{2}\right)^2 \\ &= B^2 + C^2 + D^2. \end{aligned}$$