Math 550 Oct 17, 2001
Vector Analysis
Fall 2001, USC

Solutions to Midterm 2

1.

$$\nabla \times F = \begin{vmatrix} yz & xz & xy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ i & j & k \end{vmatrix} = \Big(x - x, -(y - y), z - z\Big) = (0, 0, 0).$$

So F is a gradient vector field, because its curl vanishes everywhere.

2. If a particle moving along a path $\mathbf{c}(t)$ has constant speed, then $\|\mathbf{c}'(t)\|^2$ is constant. So we have:

$$0 = (\|\mathbf{c}'(t)\|^2)' = (\mathbf{c}'(t) \cdot \mathbf{c}'(t))' = \mathbf{c}''(t) \cdot \mathbf{c}'(t) + \mathbf{c}'(t) \cdot \mathbf{c}''(t) = 2\mathbf{c}'(t) \cdot \mathbf{c}''(t).$$

Thus $\mathbf{c}'(t) \cdot \mathbf{c}''(t) = 0$, which means that the velocity and acceleration vectors are orthogonal.

3.

$$\underset{[0,2\pi]}{\text{Length}}[\mathbf{c}] := \int_0^{2\pi} \|\mathbf{c}'(t)\|\, dt = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1}\, dt = 2\sqrt{2}\pi.$$

4.

$$\begin{aligned} \|u \times v\|^2 &= \|u\|^2\|v\|^2 \sin^2\theta = \|u\|^2\|v\|^2(1 - \cos^2\theta) \\ &= \|u\|^2\|v\|^2 - \|u\|^2\|v\|^2 \cos\theta = \|u\|^2\|v\|^2 - (u \cdot v)^2 \end{aligned}$$

5. Recall that if line passes through a point p_0 and has direction u, then its distance from a point p is given by

$$\text{dist}(p, \ell) = \frac{\|\overrightarrow{p_0 p} \times u\|}{\|u\|}.$$

In this problem, $p = (2,2,0)$, and we may set $p_0 = (2,3,1)$, and $u = (1,1,1)$. So

$$d = \frac{\|(0,1,1)\times(1,1,1)\|}{\sqrt{3}} = \frac{1}{\sqrt{3}}\left\|\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ i & j & k \end{vmatrix}\right\| = \sqrt{\frac{2}{3}}$$

6. First recall that if $\mathbf{F}$ is a vector field and f is a scalar function, then $\nabla \cdot (f\mathbf{F}) = (\nabla f)\cdot \mathbf{F} + f\nabla \cdot \mathbf{F}$. Thus

$$\nabla \cdot (\frac{1}{r^3}\mathbf{r}) = (\nabla \frac{1}{r^3})\cdot \mathbf{r} + \frac{1}{r^3}\nabla \cdot \mathbf{r}.$$

Since $1/r^3 = (x^2+y^2+z^2)^{-3/2}$,

$$\nabla \frac{1}{r^3} = (\frac{-3x}{r^5}, \frac{-3y}{r^5}, \frac{-3z}{r^5}) = -3\frac{\mathbf{r}}{r^5},$$

Further,

$$\nabla \cdot \mathbf{r} = 1+1+1 = 3.$$

So, combining the three equations above, we get

$$\nabla \cdot (\frac{1}{r^3}\mathbf{r}) = -3\frac{\mathbf{r}}{r^5}\cdot \mathbf{r} + \frac{1}{r^3}3 = \frac{-3r^2}{r^5} + \frac{3}{r^3} = 0.$$

7. a) Note that

$$h'(t) = (\mathbf{c}(t)\times \mathbf{c}'(t))' = \mathbf{c}'(t)\times\mathbf{c}'(t) + \mathbf{c}'(t)\times\mathbf{c}''(t) = 0 + \mathbf{c}'(t)\times m\mathbf{c}'(t) = 0,$$

because the cross product of parallel vectors is zero. Therefore, h is constant.

b) $\mathbf{c}(t)\cdot h(t) = \mathbf{c}(t)\cdot\mathbf{c}(t)\times\mathbf{c}'(t) = \mathbf{c}'(t)\cdot\mathbf{c}(t)\times\mathbf{c}(t) = 0$. So $\mathbf{c}(t)$ lies in the plane which passes through the origin and orthogonal to $h(t)$.

So, since $h(t)$ is constant, $\mathbf{c}(t)$ lies in a fixed plane.