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Differential Geometry
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Lecture Notes 2

1.4 Definition of Manifolds

By a *basis* for a topological space (X, T), we mean a subset B of T such that for any $U \in T$ and any $x \in U$ there exists a $V \in B$ such that $x \in V$ and $V \subset U$.

Exercise 1.4.1. Let $\mathbf{Q}$ denote the set of rational numbers. Show that

$$\{B^n_{1/m}(x) \mid x \in \mathbf{Q}^n \text{ and } m = 1, 2, 3, \dots\}$$

forms a basis for $\mathbf{R}^n$. In particular, $\mathbf{R}^n$ has a countable basis. So does any subset of $\mathbf{R}^n$ with the subspace topology.

Exercise 1.4.2. Let T be the topology on $\mathbf{R}$ generated as follows. We say that a subset U of $\mathbf{R}$ is open if for every $x \in U$, there exist a, $b \in \mathbf{R}$ such that $x \in [a, b)$ and $[a, b) \subset U$. Show that T does not have a countable basis. (*Hint:* Let B be a basis for T, and for each $x \in \mathbf{R}$, let B_x be the basis element such that $x \in B_x$ and $B_x \subset [x, x+1)$.)

A toplogical space X is said to be *Hausdorf*, if for every pair of distinct points p_1, $p_2 \in X$, there is a pair of disjoint open subsets U_1, U_2 such that $p_1 \in U_1$ and $p_2 \in U_2$.

Exercise 1.4.3. Show that any compact subset of a Hausdorf space X is closed in X.

Exercise 1.4.4. Let X be compact, Y be Hausdorf, and $f: X \to Y$ be a continuous one-to-one map. Then f is a homeomorphism between X and $f(X)$.

We say that $X \subset \mathbf{R}^n$ is *convex* if for every x, $y \in X$, the line segment

$$\lambda x + (1 - \lambda) y, \quad \lambda \in [0, 1]$$

lies in X

Exercise 1.4.5 (Topology of Convex Sets). Show that every compact convex subset of $\mathbf{R}^n$, which contains an open subset of $\mathbf{R}^n$, is homeomorphic to $B^n_1(o)$. (*Hint:* Suppose that o lies in the open set which lies in X. Define $f: \mathbf{S}^{n-1} \to \mathbf{R}$ by $f(u) := \sup_{x \in X} \langle u, x \rangle$. Show that $g: X \to B^n_1(o)$, given by $g(x) := x/f(x/\|x\|)$, if $x \neq o$, and $g(o) := o$, is a homeomorphism.)

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By a *neighborhood* of a point x of a topological space X we mean an open subset of X which contains x. We say a topological space X is *locally homeomorphic* to a topological space Y if each $x \in X$ has a neighborhood which is homeomorphic to Y.

By a *manifold* M, we mean a topological space which satisfies the following properties:

1. M is hausdorf.
2. M has a countable basis.
3. M is locally homeomorphic to $\mathbf{R}^n$.

The "n" in item 3 in the above defintion is called the *dimension* of M.

Exercise 1.4.6. Show that condition 3 in the definition of manifold may be replaced by the following (weaker) condition:

3'. For every point p of M there exist an open set $U \subset \mathbf{R}^n$ and a one-to-one continuous mapping $f: U \to M$, such that $p \in f(U)$.

Conditions 1 and 2 are not redondant, as demonstrated in the following Exercise:

Exercise 1.4.7. Let X be the union of the lines $y = 1$ and $y = -1$ in $\mathbf{R}^2$, and P be the partition of X consisting of all the subsets of the form $\{(x, 1)\}$ and $\{(x, -1)\}$ where $x \geq 0$, and all sets of the form $\{(x, 1), (x, -1)\}$ where $x < 0$. Show that X is locally homeomorphic to $\mathbf{R}$ but is not hausdorf.

It can also be shown that there exist manifolds which satisfy conditions 1 and 3 but not 2. One such example is the "long line", see Spivak.

Finally, it turns out that we do not need to wory about condition 2 if our topological space is compact

Theorem 1.4.8. *If a topological space is compact, and satisfies conditions* 1 *and* 3*, then it satisfies condition* 2 *as well. In particular, it is a manifold.*

1.5 Examples of Manifolds

Exercise 1.5.1. Show that $\mathbf{S}^n$ is a manifold.

Exercise 1.5.2. Show that any open subset of a manifold is a manifold, with respect to the subspace topology.

Exercise 1.5.3 (Product Manifolds). If M and N are manifolds of dimension m and n respectively, show that $M \times N$ is a manifold of dimension $m + n$, with respect to its product topology. In particular, the torus T^n is an n-dimensional manifold.

We say that a group G *acts* on a topological space X if for every $g \in G$ there exists a homeomorphism $f_g: X \to X$ such that

1. f_e is the indentity function on X.

2. $f_g \circ f_h = f_{g \circ h}$.

where e is the indentity element of G. For each $p \in X$, the *orbit* of p is

$$[p] := \{g(p) \mid g \in G\}.$$

Exercise 1.5.4. Show that The collection of orbits $P := \{[p] \mid p \in X\}$ is a partition of X.

When P is endowed with the quotient topology, then the resulting space is denoted as X/G.

Exercise 1.5.5. For each integer $z \in \mathbf{Z}$, let $g_z : \mathbf{R} \to \mathbf{R}$ be defined by $g_z(x) := x+z$. Show that $\mathbf{Z}$ acts on $\mathbf{R}$, and $\mathbf{R}/\mathbf{Z}$ is homeomorphic to $\mathbf{S}^1$.

Exercise 1.5.6. Define an action of $\mathbf{Z}^n$ on $\mathbf{R}^n$ so that $\mathbf{R}^n/\mathbf{Z}^n$ is homeomorphic to T^n.

Let $\pi \colon X \to X/G$ be given by

$$\pi(p) := [p].$$

We say that a mapping $f \colon X \to Y$ is open if for every open $U \subset X$, $f(U)$ is open in Y.

Exercise 1.5.7. Show that $\pi \colon X \to X/G$ is open.

We say that G acts *properly discontinuously* on X, if

1. For every $p \in X$ and $g \in G - \{e\}$ there exists a neighborhood U of p such that $U \cap g(U) = \emptyset$.

2. For every p, $q \in X$, such that $p \neq h_g(q)$ for any $g \in G$, there exist neighborhoods U and V respectively, such that $U \cap g(V) = \emptyset$ for all $g \in G$.

Exercise 1.5.8 (Group Actions). Show that if a group G acts properly discontinuosly on a manifold M, then M/G is a manifold. (*Hints:* Openness of π ensures that M/G has a countable basis. Condition (i) in the defintion of proper discontinuity ensures that π is locally one-to-one, which together with openness, yields that M/G is locally homeomorphic to $\mathbf{R}^n$. Finally, condition (ii) implies that M/G is hausdorf.)

Exercise 1.5.9. Show that $\mathbf{RP}^n$ is homeomorphic to $\mathbf{S}^n/\{\pm 1\}$, so it is a manifold.

Exercise 1.5.10 (Hopf Fibration). Note that, if $\mathbf{C}$ denotes the complex plane, then $\mathbf{S}^1 = \{z \in \mathbf{C} \mid \|z\| = 1\}$. Thus, since $\|zw\| = \|z\|\,\|w\|$, $\mathbf{S}^1$ admits a natural group structure. Further, note that $\mathbf{S}^3 = \{(z_1, z_2) \mid \|z_1\|^2 + \|z_2\|^2 = 1\}$. Thus, for every $w \in \mathbf{S}^1$, we may define a mapping $f_w \colon \mathbf{S}^3 \to \mathbf{S}^3$ by $f_w(z_1, z_2) := (wz_1, wz_2)$. Show that this defines a group action on $\mathbf{S}^3$, and $\mathbf{S}^3/\mathbf{S}^1$ is homeomorphic to $\mathbf{S}^2$.

Exercise 1.5.11 (Piecewise Linear (PL) manifolds). Suppose that we have a collection X of triangles, such that (i) each edge of a triangle in X is shared by exactly one other triangle (ii) whenever two triangles of X intersect, they intersect at a common vertex or along a common edge, (iii) each subset of X consisting of all the triangles which share a vertex is finite and remains connected, if that vertex is deleted. Show that X is a 2-dimensional manifold.

The converse of the problem in the above exercise is also true: every two dimensional manifold can be "triangulated".

Exercise 1.5.12. Generalize the previous exercise to 3-dimensional manifolds.

1.6 Classification of Manifolds

The following theorem is not so hard to prove, though it is a bit tediuos, specially in the noncompact case:

Theorem 1.6.1. *Every connected* 1*-dimensional manifold is homeomorphic to either* $\mathbf{S}^1$*, if it is compact, and to* $\mathbf{R}$ *otherwise.*

To describe the classification of 2-manifolds, we need to introduce the notion of *connected sums*. Let M_1 and M_2 be a pair of n-dimensional manifolds and X_1 and X_2 be subsets of M_1 and M_2 respectively, which are homeomorphic to the unit ball $B^n_1(o)$. Let Y_1 and Y_2 be subsets of X_1 and X_2 which are homeomorphic to the open ball $U^n_1(o)$, and set

$$X := (M_1 - Y_1) \cup (M_2 - Y_2).$$

Let $\partial X_1 := X_1 - Y_1$ and $\partial X_2 := X_2 - Y_2$. The each of ∂X_1 and ∂X_2 are homeomorphic to $\mathbf{S}^n$. In particular, there exist a homeomorphism $f\colon \partial X_1 \to \partial X_2$. Let P be the partition of X consiting of all single sets of points in $X - (\partial X_1 \cup \partial X_2)$ and all sets of the form $\{p, f(p)\}$. The resulting quotient space is called the *connected sum* of M_1 and M_2.

Theorem 1.6.2. *Every compact connected* 2 *dimensional manifold is homeomorphic to exactly one of the following:* $\mathbf{S}^2$*,* $\mathbf{RP}^2$*,* T^2*, the connected sum of finitely many* T^2*s, or the connected sum of finitely many* $\mathbf{RP}^2$*s.*

Exercise 1.6.3 (The Klein Bottle). Show that the connected sum of two $\mathbf{RP}^2$s is homeomorphic to the quotient space obtained from the following partition P of $[0, 1] \times [0, 1]$: P consists of all the single sets $\{(x, y)\}$ where $(x, y) \in (0, 1) \times (0, 1)$, all sets of the form {(x,0),(x,1)} where $x \in (0, 1)$, and all the sets of the form $\{(0, y), (1, 1 - y)\}$ where $y \in [0, 1]$

Once the dimension reaches 3, however, comparitively very little is known about classification of manifolds, and in dimensions 5 and higher it can be shown that it would not be possible to devise any sort of algorithm for such classifications. Thus it is in dimensions 3 and 4 where the topology of manifolds is of the most interest.

The outstanding question in 3-manifold topology, and perhaps in all of Mathematics (the only other problem which one might claim to have priority is Riemann's hypothesis) is the *Poincare's conjecture*, which we now describe.

We say that a manifold M is *simply connected* if for every continuous mapping $f\colon \mathbf{S}^1 \to M$ there is a continuous mapping $g\colon B_1^2(o) \to M$, such that $g = f$ on $\mathbf{S}^1$. For instance, it is intuitively clear that $\mathbf{S}^2$ is simply connected, but T^2 is not.

Problem 1.6.4 (Poincare's Conjecture). Prove that every compact connected and simply connected 3-dimensional manifold is homeomorphic to $\mathbf{S}^3$.

The generalizations of the above problem to dimesions 5 and higher have been solved by Smale, and in dimension 4, by Freedman, both of whom won the fields medal. Ironically enough, however, Poincare proposed his conjecture only in dimension 3.

The above problem is now one of the Clay Mathematical Institute's "millenial prize problems", that is, there is a one million dollar reward for solving Poincare's conjecture (not to mention a Fields medal and a host of other accolades).

1.7 Manifolds with boundary

The Euclidean *upper half space* is define by

$$\mathbf{H}^n := \{\, (x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_n > 0 \,\}.$$

By a *manifold-with boundary*, we mean a hausdorf topological space, with a countable basis, which is locally homeomprphic to either $\mathbf{R}^n$ or $\mathbf{H}^n$. If M is a manifold-with-boundary, then the boundary of M, denoted by ∂M, is defined as the set of all points p of M such that no neighborhood of p is homeomorphic to $\mathbf{R}^n$.

Exercise 1.7.1 (Boundary of a boundary). Show that if M is an n-dimensional manifold-with-boundary, then ∂M is an $(n-1)$-dimensional manifold (without boundary).

Exercise 1.7.2 (Double of a manifold). Show that every n dimensional manifod-with-boundary lies in an n dimensional manifold.

Exercise 1.7.3 (The Mobius Strip). Let $X := [1, 0] \times [1, 0]$ and let P be the partition of X consisting of all sinlge sets $\{(x, y)\}$ where $x \in (0, 1)$ and all the sets of the form $\{(0, y), (0, 1 - y)\}$ where $y \in [0, 1]$. Show that P is a manifold-with-boundary with respect to the quotient topology.

Let M_1 and M_2 be a pair of manifolds-with-boundary and suppose that ∂M_1 is homeomorphic to ∂M_2. Let $f\colon \partial M_1 \to \partial M_2$ be a homeomorphism and set $X :=$

$M_1 \cup M_2$. Let P be the partition of X consisting of all single sets $\{x\}$ where $x \in X - (\partial M_1 \cup \partial M_2)$ and all sets of the form $\{x, f(x)\}$ where $x \in \partial M_1$. Then P, with its quotient toplogy, is called a *gluing* of M_1 and M_2.

Exercise 1.7.4 (Gluing of manifolds-with-boundary). Show that gluing of two manifold M_1 and M_2 with boundary yields a manifold without boundary, and this manifold is independent of the choice of the homeomorphisms $f : \partial M_1 \to \partial M_2$.

Exercise 1.7.5 (Mobius and Klein). Show that the gluing of two Mobius strips yields a Klein Bottle.

The classifications of 2 dimensional manifolds with boundary is well understood:

Theorem 1.7.6. *Every compact connected* 2 *dimensional manifold-with-boundary is homeomorphic to a* 2 *dimensional manifold from which a finite number of subsets each homeomorphic to an open ball has been removed.*

For the following exercise assume:

Theorem 1.7.7 (Generalized Jordan-Brouwer). *Let M be a connected n dimensional manifold, and N be a subset of M which is homeomorphic to a compact connected $n-1$ dimensional manifold. Them $M - N$ has exactly two components.*

Exercise 1.7.8 (Alternative form of Poincare). Show that Poincare's conjecture is equivalent to the following: the only compact connected and simply connected manifold-with-boundary whose boundary is homeomorphic to $\mathbf{S}^2$ is homeomorphic to the ball $B_1^3(o)$.