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Differential Geometry I
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# Lecture Notes 12

## Riemannian Metrics

### 0.1 Definition

If $M$ is a smooth manifold then by a Riemannian metric $g$ on $M$ we mean a smooth assignment of an innerproduct to each tangent space of $M$. This means that, for each $p \in M$, $g_p \colon T_pM \times T_pM \to \mathbf{R}$ is a symmetric, positive definite, bilinear map, and furthermore the assignment $p \mapsto g_p$ is smooth, i.e., for any smooth vector fields $X$ and $Y$ on $M$, $p \mapsto g_p(X_p, Y_p)$ is a smooth function. The pair $(M, g)$ then will be called a Riemannian manifold. We say that a diffeomorphism $f \colon M \to N$ between a pair of Riemannian manifolds $(M, g)$ and $(N, h)$ is an *isometry* provided that

$$g_p(X, Y) = h_{f(p)}(df_p(X), df_p(Y))$$

for all $p \in M$ and $X$, $Y \in T_pM$.

**Exercise 0.1.1.** Show that the antipodal reflection $a \colon \mathbf{S}^n \to \mathbf{S}^n$, $a(x) := -x$ is an isometry.

### 0.2 Examples

#### 0.2.1 The Euclidean Space

The simplest example of a Riemannian manifold is $\mathbf{R}^n$ with its standard Euclidean innerproduct, $g(X, Y) := \langle X, Y\rangle$.

#### 0.2.2 Submanifolds of a Riemannian manifold

A rich source of examples are generated by immersions $f \colon N \to M$ of any manifold $N$ into a Riemannian manifold $M$ (with metric $g$); for this induces a metric $h$ on $N$ given by

$$h_p(X, Y) := g_{f(p)}(df_p(X), df_p(Y)).$$

In particular any manifold may be equipped with a Riemannian metric since every manifold admits an embedding into $\mathbf{R}^n$.

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### 0.2.3 Quotient of a Riemannian manifold by a group of isometries

Note that the set of isometries $f\colon M \to M$ forms a group. Another source of examples of Riemannian manifolds are generated by taking the quotient of a Riemannian manifold $(M, g)$ by a subgroup $G$ of its isometries which acts properly discontinuously on $M$. Recall that if $G$ acts properly discontinuously, then $M/G$ is indeed a manifold. Then we may define a metric $h$ on $M/G$ by setting $h_{[p]} := g_p$. More precisely recall that the projections $\pi\colon M \to M/G$, given by $\pi(p) := [p]$ is a local diffeomorphism, i.e., for any $q \in [p]$ there exists an open neighborhood $U$ of $p$ in $M$ and an open neighborhood $V$ of $[p]$ in $M/G$ such that $\pi\colon U \to V$ is a diffeomorphism. Then we may define

$$h_{[p]}(X, Y) := g_q((d\pi_q)^{-1}(X), (d\pi_q)^{-1}(Y)).$$

One can immediately check that $h$ does not depend on the choice of $q \in [p]$ and is thus well defined.A specific example of proper discontinuous action of isometries is given by translations $f_z\colon \mathbf{R}^n \to \mathbf{R}^n$ given by $f_z(p) := p + z$ where $z \in \mathbf{Z}^n$. Recall that $\mathbf{R}^n/\mathbf{Z}^n$ is the torus $T^n$, which may now be equipped with the metric induced by this group action. Similarly $\mathbf{RP}^n$ admits a canonical metric, since $\mathbf{RP}^n = \mathbf{S}^n/\{\pm 1\}$, and reflections of a sphere are isometries.

### 0.2.4 Conformal transformations

As another set of examples note that if $(M, g)$ is any Riemannian manifold, then $(M, \lambda g)$ is also a Riemannian manifold where $\lambda\colon M \to \mathbf{R}^+$ is any smooth positive function. Note that this change of metric does not effect the angles between any pair of vectors in a tangent space of $M$. Thus $(M, \lambda g)$ is said to be *conformal* to $(M, g)$.

**Exercise 0.2.1.** Show that the inversion $i\colon \mathbf{R}^n - \{o\} \to \mathbf{R}^n$ given by $i(x) := x/\|x\|^2$ is a conformal transformation.

**Exercise 0.2.2.** Show that the stereographic projection $\pi\colon \mathbf{S}^2 - \{(0, 0, 1)\} \to \mathbf{R}^2$ is a conformal transformation.

### 0.2.5 The hyperbolic space

Finally, an important example is the hyperbolic space which may be represented by a number of models. One model, known as Poincare's half space model, is to take the open upper half space of $\mathbf{R}^n$ and define there a metric via

$$g_p(X, Y) := \frac{\langle X, Y \rangle}{(p_n)^2},$$

where $p_n$ denotes the $n^{th}$ coordinate of $p$. Another description of the hyperbolic space may be given by taking the open unit ball if $\mathbf{R}^n$ and defining

$$g_p(X,Y) := \frac{\langle X, Y\rangle}{(1-\|p\|^2)^2}.$$

This is known as Poincare's ball model.

**Exercise 0.2.3.** Show that the the Poincare half-plane and the half-disk are isometric (Hint: identify the Poincare half-plane with the region $y > 1$ in $\mathbf{R}^2$ and do an inversion).

## 0.3 Metric in local coordinates

Let $(U, \phi)$ be a local chart for $(M, g)$. Then, recall that if $e_1, \dots e_n$ denote the standard basis of $\mathbf{R}^n$, we obtain a basis for each $T_pM$, for $p \in U$ by setting

$$E_i(p) := d\phi^{-1}_{\phi(p)}(e_i).$$

Now if $X, Y \in T_pM$, then $X = \sum_{i=1}^n X^i E_i$ and $Y = \sum_{i=1}^n Y^i E_i$. Further, if we set

$$g_{ij}(p) := g_p(E_i, E_j),$$

then, since $g$ is bilinear we have

$$g_p(X,Y) = \sum_{i,j=1}^n X^i Y^j g_p(E_i, E_j) = \sum_{i,j=1}^n X^i Y^j g_{ij}(p).$$

Thus in any local coordinate $(U, \phi)$ a metric is completely determined by the functions $g_{ij}$ which may be regarded as the coefficients of a positive definite matrix.

To obtain a concrete example, note that if $M \subset \mathbf{R}^n$ is a submanifold, with the induce metric from $\mathbf{R}^n$, and $(\phi, U)$ is a local chart of $M$, then if we set $f := \phi^{-1}$, $f\colon \phi(U) \to \mathbf{R}^n$ is a *parametrization* for $U$, and $d(f)(e_i) = D_i f$. Consequently,

$$g_{ij}(p) = \langle D_i f(f^{-1}(p)), D_j f(f^{-1}(p))\rangle.$$

For instance, note that a surface of revolution in $\mathbf{R}^3$ which is given by rotating the curve $(r(t), z(t))$ in the $xz$-plane about the $z$ axis can be parametrized by

$$f(t,\theta) = (r(t)\cos\theta, r(t)\sin\theta, z(t)).$$

So

$$D_1 f(t,\theta) = (r'(t)\cos\theta, r'(t)\sin\theta, z'(t)) \quad \text{and} \quad D_2 f(t,\theta) = (-r(t)\sin\theta, r(t)\cos\theta, 0),$$

and consequently $g_{ij}(f(t,\theta))$ is given by

$$\begin{pmatrix} (r')^2 + (z')^2 & 0 \\ 0 & r^2 \end{pmatrix}.$$

Note that if we assume that the curve in the $xz$-plane is parametrized by arclength, then $(r')^2 + (z')^2 = 1$, so the above matrix becomes more simple to work with.

**Exercise 0.3.1.** Compute the metric of $\mathbf{S}^2$ in terms of spherical coordinates $\theta$ and $\phi$.

**Exercise 0.3.2.** Compute the metric of the surface given by the graph of a function $f\colon \Omega \subset \mathbf{R}^2 \to \mathbf{R}$.

## 0.4 Length of Curves

In a Riemannian manifold $(M, g)$, the length of any piecewise smooth curves $c\colon [a, b] \to M$ with $c(a) = p$ and $c(b) = q$ is defined as

$$\text{Length}[c] := \int_a^b \sqrt{g_{c(t)}(c'(t), c'(t))}\, dt,$$

where

$$c'(t) := dc_t(1).$$

Note that the definition for the length of curves here is a generalization of the Euclidean case where we integrate the speed of the curve. Indeed the last formula above coincides with the regular notion of derivative when $M$ is just $\mathbf{R}^n$. To see this, recall that $dc_t(1) = (c \circ \gamma)'(0)$ where $\gamma\colon (\epsilon, \epsilon) \to [a, b]$ is a curve with $\gamma(0) = t$ and $\gamma'(0) = 1$, e.g., $\gamma(u) = t+u$. Thus by the chain rule $(c\circ\gamma)'(0) = c'(\gamma(0))\gamma'(0) = c'(t)$.

**Exercise 0.4.1.** Compute the length of the radius of the Poincare-disk (with respect to the Poincare metric).

## 0.5 The classical notation for metric

For any curve $c\colon [a, b] \to \mathbf{R}^n$ we may write $c(t) = (x_1(t), \dots, x_n(t))$. Consequently, if we define $g_{ij}(p) := g_p(e_i, e_j)$ where $e_1, \dots, e_n$ is the standard basis for $\mathbf{R}^n$, then bilinearity of $g$ yields that

$$g_{c(t)}(c'(t), c'(t)) = \sum_{i,j=1}^{n} g_{c(t)}(e_i, e_j)x_i'(t)x_j'(t) = \sum_{i,j=1}^{n} g_{ij}(c(t))x_i'(t)x_j'(t).$$

Thus we may write

$$\text{Length}[c] := \int_a^b \sqrt{\sum_{i,j=1}^{n} g_{ij}(c(t))\frac{dx_i}{dt}\frac{dx_j}{dt}}\, dt.$$

Indeed classically metrics were specified by an expression of the form

$$ds^2 = \sum_{i,j=1}^{n} g_{ij} dx_i dx_j.$$

and then length of a curve was defined as the integral of $ds$, which was called "the element of arclength", along that curve:

$$\text{Length}[c] = \int_c ds.$$

In particular note that, in the classical notation, the standard Euclidean metric in the plane is given by $ds^2 = \sum_{i=1}^n dx_i^2$. Further, in the Poincare's half-disk model, $ds^2 = \sum_{i=1}^n dx_i^2/x_n^2$.

## 0.6 Distance

For any pairs of points $p$, $q \in M$, let $C(p,q)$ denote the space piecewise smooth curves $c\colon [a,b] \to M$ with $c(a) = p$ and $c(b) = q$. Then, if $M$ is connected, we may define the distance between $p$ and $q$ as

$$d_g(p,q) := \inf\{\text{Length}[c] \mid c \in C(p,q)\}.$$

So the distance between a pair of points is defined as the greatest lower bound of the lengths of curves which connect those points. First we show that this is a generalization of the standard notion of distance in $\mathbf{R}^n$.

**Lemma 0.6.1.** *For all continuous maps $f\colon (a,b) \to \mathbf{R}^n$*

$$\left\| \int_a^b f(t)dt \right\| \leq \int_a^b \|f(t)\| dt.$$

*Proof.* By the Cauchy-Schwarts inequality, for any unit vector $u \in \mathbf{S}^{n-1}$,

$$\left\langle \int_a^b f(t)dt, u \right\rangle = \int_a^b \langle f(t), u\rangle dt \leq \int_a^b \|f(t)\| dt.$$

In particular we may let $u := \int_a^b f(t)dt / \| \int_a^b f(t)dt\|$, assuming that $\int_a^b f(t)dt \neq 0$ (otherwise the lemma is obviously true). □

**Corollary 0.6.2.** *If $(M,g) = (\mathbf{R}^n, \langle\rangle)$ then $d_g(p,q) = \|p-q\|$.*

*Proof.* First note that if we set $c(t) := (1-t)p + tq$, then

$$\text{Length}[c] := \int_0^1 \|p-q\| \, dt = \|p-q\|.$$

So $d_g(p,q) \leq \|p-q\|$. It remains then to show that $d_g(p,q) \geq \|p-q\|$. The later inequality holds because for all curves $c\colon [a,b] \to \mathbf{R}^n$

$$\int_a^b \|c'(t)\| \, dt \geq \left\| \int_a^b c'(t)dt \right\| = \|c(b) - c(a)\|.$$

□

The previous result shows that $(M, d_g)$ is a metric space when $M$ is the Euclidean space $\mathbf{R}^n$ and $g$, which induces $d$, is the standard innerproduct. Next we show that this is the case for all Riemannian manifolds. To this end we first need a local lemma:

**Lemma 0.6.3.** *Let $(B, g)$ be a Riemannian manifold, where $B := \overline{B}^n_r(o) \subset \mathbf{R}^n$. Then there exists $m > 0$ such that for any piecewise $C^1$ curve $c\colon [a,b] \to B$ with $c(a) = o$ and $c(b) \in \partial B$ we have* Length$[c] > m$.

*Proof.* Define $f\colon \mathbf{S}^{n-1} \times B \to \mathbf{R}$ by $f(u,p) := g_p(u,u)$. Note that, since $g$ is positive definite, $f > 0$. Thus since $f$ is continuous and $\mathbf{S}^{n-1} \times B$ is compact $f \geq \lambda^2 > 0$. Consequently, bilinearity of $g$ yields that

$$g_p(v,v) \geq \lambda^2 \|v\|^2.$$

The above inequality is obvious when $\|v\| = 0$, and when $\|v\| \neq 0$, observe that $g_p(v,v) = g_p(v/\|v\|, v/\|v\|)\|v\|^2$. Next note that

$$\mathrm{Length}[c] = \int_a^b \sqrt{g_{c(t)}(c'(t), c'(t))}\, dt \geq \lambda \int_a^b \|c'(t)\|\, dt.$$

But $\int_a^b \|c'(t)\|$ is just the length of $c$ with respect to the standard metric on $\mathbf{R}^n$. Thus, by the previous proposition,

$$\int_a^b \|c'(t)\|\, dt \geq \|c(b) - c(a)\| = r.$$

So setting $m := \lambda r$ finishes the proof. □

The proof of the next observation is immediate:

**Lemma 0.6.4.** *If $f\colon M \to N$ is an isometry, then* Length$[c]$ = Length$[f \circ c]$ *for any piecewise $C^1$ curve $c\colon [a,b] \to M$.*

□

Note that if $(M, g)$ is a Riemannian manifold and $f\colon M \to N$ is a diffeomorphism between $M$ and any smooth manifold $N$, then we may *push forward* the metric of $M$ by defining

$$df(g)_p(X,Y) := g_{f^{-1}(p)}(df^{-1}(X), df^{-1}(Y)).$$

Then $f$ is an isometry between $(M, g)$ and $(N, df(g))$. In particular we may assume that any local charts $(U, \phi)$ on a Riemannian manifold $(M, g)$ is an isometry, with respect to the push forward metric $d\phi(g)$ on $\phi(U)$. This observation, together with the previous lemma easily yields that:

**Proposition 0.6.5.** *If $(M, g)$ is any Riemannian manifold then $(M, d_g)$ is a metric space.*

*Proof.* It is immediate that $d$ is symmetric and satisfies the triangle inequality. Furthermore it is clear that $d$ is always nonnegative. Showing that $d$ is positive definite, however, requires more work. Specifically, we need to show that when $p \neq q$, then $d(p,q) > 0$. Suppose $p \neq q$. Then, since $M$ is Hausdorff, there exists an open neighborhood $V$ of $p$ such that $q \notin V$. Let $(U, \phi)$ be a local chart centered at $p$. Choose $r$ so small that $\overline{B}_r(o) \subset \phi(V \cap U)$, and set $W := \phi^{-1}(\overline{B}_r(o))$. Then $\phi \colon W \to \overline{B}_r(o)$ is a diffeomorphism, and we may equip $\overline{B}_r(o)$ with the push forward metric $d\phi(g)$ which will turn $\phi$ into an isometry. Now let $c \colon [a,b] \to M$ be any piecewise $C^1$ curve with $c(a) = p$ and $c(b) = q$. Then there exist $a \leq b' \leq b$ such that $c[a,b'] \subset W$ and $c(b') \in \partial W$ (to find $b'$ let $\overset{\circ}{W} := \phi^{-1}(B_r(o))$ be the interior of $W$, then $c^{-1}(\overset{\circ}{W})$ is an open subset of $[a,b]$ which contains $a$, and we may let $b'$ be the upperbound of the component of $c^{-1}(\overset{\circ}{W})$ which contains $a$.) Let $\overline{c} \colon [a,b'] \to W$ be the restriction of $c$. Then obviously $\mathrm{Length}[c] \geq \mathrm{Length}[\overline{c}]$. But $\mathrm{Length}[\overline{c}] = \mathrm{Length}[\phi \circ \overline{c}]$ since $\phi$ is an isometry, and by the previous lemma then length of any curve in $(B_r^n(o), d\phi(g))$ which begins at the center of the ball and ends at its boundary is bounded below by a positive constant. $\square$

Now recall that any metric space has a natural topology. In particular $(M, d_g)$ is a topological space. Next we show that this topological space is identical to the original $M$.

**Lemma 0.6.6.** *Let $(M, g^1)$, $(M, g^2)$ be Riemannian manifolds, and suppose $M$ is compact. Then there exist a constant $\lambda > 0$ such that for any $p$, $q \in M$ we have*

$$d_{g^1}(p,q) \geq \lambda\, d_{g^2}(p,q).$$

*Proof.* Define $f \colon \mathbf{S}^{n-1} \times M \to \mathbf{R}$ by $f(u,p) := g^1_p(u,u)/g^2_p(u,u)$. Note that, since $g$ is positive definite, $f > 0$. Thus since $f$ is continuous and $\mathbf{S}^{n-1} \times M$ is compact $f \geq \lambda^2 > 0$. Consequently, bilinearity of $g$ yields that

$$g^1_p(v,v) \geq \lambda^2\, g^2_p(v,v),$$

for all $v \in \mathbf{R}^n$. Next note that the above inequality yields

$$\mathrm{Length}_{g_1}[c] = \int_a^b \sqrt{g^1_{c(t)}(c'(t), c'(t))}\, dt \geq \lambda \int_a^b \sqrt{g^2_{c(t)}(c'(t), c'(t))}\, dt = \lambda\, \mathrm{Length}_{g_2}[c].$$

for any curve $c \colon [a,b] \to M$. In particular the above inequalities hold for all curves $c \colon [a,b] \to M$ with $c(a) = p$ and $c(b) = q$. $\square$

**Proposition 0.6.7.** *The metric space $(M, d_g)$, endowed with its metric topology, is homeomorphic to $M$ with its standard topology.*

*Proof.* There are two parts to this argument:

Part I: We have to show that every open neighborhood $U$ of $M$ is open in its metric topology, i.e., for every $p \in U$ there exists an $r > 0$ such that $B_r^g(p) \subset U$, where

$$B_r^g(p) := \{q \in M \mid d_g(p,q) < r\}.$$

To see this first note that, as we showed in the proof of the previous proposition, there exists an open neighborhood $V$ of $p$ with $V \subset U$ such that there exists a homeomorphism $\phi \colon \overline{V} \to \overline{B}_1^n(o)$. Now, much as in the proof of the previous proposition, if we endow $\overline{B}_1^n(o)$ with the push forward metric induced by $\phi$ then $(\overline{B}_1^n(o), d\phi(g))$ becomes isometric to $(\overline{V}, g)$. But recall that, as we showed in the earlier proposition, the distance of any point in the boundary $\partial B_1^n(o) = \mathbf{S}^n$ of $\overline{B}_1^n(o)$ from the origin $o$ was bigger than some constant, say $\lambda$. Thus the same is true of the distance of $\partial V$ from $p$. In particular, if we choose $r < \lambda$, then $B_r^g(p) \subset V \subset U$.

Part II: We have to show that every metric ball $B_r^g(p)$ is open in $M$, i.e., at every $q \in B_r^g(p)$ we can find a open neighborhood $U$ of $q$ in $M$ such that $U \subset B_r^g(p)$. To see this let $V$ be an open neighborhood of $p$ such that there exists a homeomorphism $\psi \colon \overline{V} \to \overline{B}_1^n(o)$, and endow $\overline{B}_1^n(o)$ with the push forward metric $d\psi(g)$. Then the distance of $\psi(\overline{V} \cap B_r^g(p))$ from $o$ is equal to $r$, with respect to the metric $d\psi(g)$. So, by the previous proposition, this distance, with respect to the Euclidean metric on $\overline{B}_1^n(o)$ must be at least $\lambda r > 0$. Thus if we choose $r' < \lambda_r$, then the Euclidean ball $B_{r'}^n(o) \subset \psi(V)$. Consequently, $U := \psi^{-1}(B_{r'}^n(o)) \subset V$, and $U$ is open in $M$, since $B_{r'}^n(o)$ is open. $\square$