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Differential Geometry I
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Lecture Notes 13

Integration on Manifolds, Volume, and Partitions of Unity

Suppose that we have an orientable Riemannian manifold (M, g) and a function $f\colon M \to \mathbf{R}$. How can we define the integral of f on M? First we answer this question locally, i.e., if (U, ϕ) is a chart of M (which preserves the orientation of M), we define

$$\int_U f dv_g := \int_{\phi(U)} f(\phi^{-1}(x))\sqrt{\det(g_{ij}^{\phi}(\phi^{-1}(x)))}dx,$$

where g_{ij} are the coefficients of the metric g in local coordinates (U, ϕ). Recall that

$$g_{ij}^{\phi}(p) := g(E_i^{\phi}(p), E_j^{\phi}(p)), \quad \text{where} \quad E_i^{\phi}(p) := d\phi^{-1}_{\phi(p)}(e_i).$$

Now note that if (V, ψ) is any other (orientation preserving) local chart of M, and $W := U \cap V$, then there are two ways to compute $\int_W f dv_g$, and for these to yield the same answer we need to have

$$\int_{\phi(W)} f(\phi^{-1}(x))\sqrt{\det(g_{ij}^{\phi}(\phi^{-1}(x)))}dx = \int_{\psi(W)} f(\psi^{-1}(x))\sqrt{\det(g_{ij}^{\psi}(\psi^{-1}(x)))}dx. \tag{1}$$

To check whether the above expression is valid recall that the change variables formula tells that if $D \subset \mathbf{R}^n$ is an open subset, $f\colon D \to \mathbf{R}$ is some function, and $u\colon \overline{D} \to D$ is a diffeomorphism, then

$$\int_D f(x)\, dx = \int_{\overline{D}} f(u(x)) \det(du_x) dx.$$

Now recall that, by the definition of manifolds, $\phi \circ \psi^{-1}\colon \psi(W) \to \phi(W)$ is a diffeomorphism. So, by the change of variables formula, the integral on the left hand side of (1) may be rewritten as

$$\int_{\psi(W)} f(\psi^{-1}(x))\sqrt{\det(g_{ij}^{\phi}(\psi^{-1}(x)))}\det(d(\phi \circ \psi)_x^{-1})dx.$$

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So for equality in (1) to hold we just need to check that

$$\sqrt{\det(g^{\psi}_{ij}(\psi^{-1}(x)))} = \sqrt{\det(g^{\phi}_{ij}(\psi^{-1}(x)))}\,\det(d(\phi\circ\psi^{-1})_x),$$

for all $x \in \psi(W)$ or, equivalently,

$$\sqrt{\det(g^{\psi}_{ij}(p))} = \sqrt{\det(g^{\phi}_{ij}(p))}\,\det(d(\phi\circ\psi^{-1})_{\psi(p)}), \tag{2}$$

for all $p \in W$. To see that the above equality holds, let (a_{ij}) be the matrix of the linear transformation $d(\phi\circ\psi^{-1})$ and note that

$$\begin{aligned} g^{\psi}_{ij} &= g(d\psi^{-1}(e_i), d\psi^{-1}(e_j)) \\ &= g(d\phi^{-1}\circ d(\phi\circ\psi^{-1})(e_i), d\phi^{-1}\circ d(\phi\circ\psi^{-1})(e_j)) \\ &= g\left(d\phi^{-1}\Big(\sum_{\ell} a_{i\ell}e_{\ell}\Big), d\phi^{-1}\Big(\sum_{k} a_{jk}e_k\Big)\right) \\ &= \sum_{\ell k} a_{il}a_{jk}g^{\phi}_{\ell k}. \end{aligned}$$

So if (g^{ψ}_{ij}) and (g^{ϕ}_{ij}) denote the matrices with the coefficients g^{ψ}_{ij} and g^{ϕ}_{ij}, then we have

$$(g^{\psi}_{ij}) = (a_{ij})(a_{ij})(g^{\phi}_{ij}).$$

Taking the determinant of both sides of the above equality yields (2). In particular note that $\sqrt{\det(a_{ij})^2} = |\det(a_{ij})| = \det(a_{ij})$, because, since M is orientable and ϕ and ψ are by assumption orientation preserving charts, $\det(a_{ij}) > 0$.

Next we discuss, how to integrate a function on all of M. To see this we need the notion of *partition of unity* which may be defined as follows: Let U_i, $i \in I$, be an open cover of M, then by a (smooth) partition of unity subordinate to U_i we mean a collection of smooth functions $\theta_i\colon M \to \mathbf{R}$ with the following properties:

1. $\operatorname{supp}\theta_i \subset A_i$.
2. for any $p \in M$ there exists only finitely many $i \in I$ such that $\theta_i(p) \neq 0$.
3. $\sum_{i\in I}\theta_i(p) = 1$, for all $p \in M$.

Here supp denotes *support*, i.e., the closure of the set of points where a given function is nonzero. Further note that by property 2 above, the sum in item 3 is well-defined.

Theorem 0.1. *If M is any smooth manifold, then any open covering of M admits a subordinate smooth partition of unity.*

Using the above theorem, whose proof we postpone for the time being, we may define $\int_M f dv_g$, for any function $f\colon M \to \mathbf{R}$ as follows. Cover M by a family of local charts (U_i, ϕ_i), and let θ_i be a subordinate partition of unity. Then we set

$$\int_M f dv_g := \sum_{i\in I} \int_{U_i} \theta_i f dv_g.$$

Note that this defintion does not depend on the choice of local charts or the corresponding partitions of unity. The *volume* of any orientable Riemannian manifold may now be defined as the integral of the constant function one:

$$\mathrm{vol}(M) := \int_M dv_g.$$

Now we proceed towards proving Theorem 0.1.

Exercise 0.2. Compute the area of a torus of revolution in $\mathbf{R}^3$.

Lemma 0.3. *Any open cover of a manifold has a countable subcover.*

Proof. Suppose that U_i, $i \in I$, is an open covering of a manifold M (where I is an arbitrary set). By defintion, M has a countable basis $B = \{B_j\}_{j\in J}$. For every $i \in I$, let $A_i := \{B_j \mid B_j \subset U_i\}$. Then A_i is an open covering for M. Next, let $A := \cup_{i\in I} A_i$. Since $A \subset B$, A is countable, so we may denote the elements of A by A_k, where $k = 1, 2, \ldots$. Note that A_k is still an open covering for M. Further, for each k there exists an $i \in I$ such that $A_k \subset U_i$. We may collect all such U_i and reindex them by k, which gives the desired countable subcover. □

Lemma 0.4. *Any manifold has a countable basis such that each basis element has compact closure.*

Proof. By the previous lemma we may cover any manifold M by a countable collection of charts (U_i, ϕ_i). Let V_j be a countable basis of $\mathbf{R}^n$ such that each V_j has compact closure $\overline{V}_j$, e.g., let V_j be the set of balls in $\mathbf{R}^n$ centered at rational points and with rational radii less than 1. Then $B_{ij} := \phi_i^{-1}(V_j)$ gives a countable basis for U_i such that each basis element has compact closure, since $\overline{B}_{ij} = \phi_i^{-1}(\overline{V}_j)$. So $\cup_{ij} B_{ij}$ gives the desired basis, since a countable collection of countable sets is countable. □

Lemma 0.5. *Any manifold M is countable at infinity, i.e., there exists a countable collection of compact subsets K_i of M such that $M \subset \cup_i K_i$ and $K_i \subset \mathrm{int}\, K_{i+1}$.*

Proof. Let B_i be the countable basis of M given by the previous lemma, i.e., with each $\overline{B}_i$ compact. Set $K_1 := \overline{B}_1$ and let $K_{i+1} := \cup_{j=1}^r \overline{B}_j$, where r is the smallest integer such that $K_i \subset \cup_{j=1}^r B_j$. □

By a *refinement* of an open cover U_i of M we mean an open cover V_j such that for each $j \in J$ there exists $i \in I$ with $V_j \subset U_i$. We say that an open covering is locally finite, if for every $p \in M$ there exists finitely many elements of that covering which contain p.

Lemma 0.6. *Any open covering of a manifold M has a countable locally finite refinement by charts (U_i, ϕ_i) such that $\phi_i(U_i) = B^n_3(o)$ and $V_i := \phi^{-1}(B^n_1(o))$ also cover M.*

Proof. First note that for every point $p \in M$, we may find a local chart (U_p, ϕ_p) such that $\phi_p(U_p) = B^n_3(o)$, and set $V_p := \phi^{-1}(B^n_1(o))$. Further, we may require that U_p lies inside any given open set which contains p. Let A_α be an open covering for M. By a previous lemma, after replacing A_α by a subcover, we may assume that A_α is countable. Now consider the sets $A_\alpha \cap (\operatorname{int} K_{i+2} - K_{i-1})$. Since $K_{i+1} - \operatorname{int} K_i$ is compact, there exists a finite number of open sets $U^{\alpha,i}_{p_j} \subset A_\alpha \cap (\operatorname{int} K_{i+2} - K_{i-1})$ such that $V^{\alpha,i}_{p_j}$ covers $A_\alpha \cap (K_{i+1} - \operatorname{int} K_i)$. Since K_i and A_α are countable, the collection $U^{\alpha,i}_{p_j}$ is a countable. Further, by construction $U^{\alpha,i}_{p_j}$ is locally finite, so it is the desired refinement. □

Note 0.7. The last result shows in particular that every manifold is *paracompact*, i.e., every open cover of M has a locally finite refinement.

Proof of Theorem 0.1. Let A_α be an open cover of M. Note that if U_i is any refinement of A_α and θ_i is a partition of unity subordinate to U_i then, θ_i is subordinate to A_α. In particular, it is enough to show that the refinement U_i given by the previous lemma has a subordinate partition of unity. To this end note that there exists a smooth nonnegative function $f\colon \mathbf{R} \to \mathbf{R}$ such that $f(x) = 0$ for $x \geq 2$, and $f(x) = 1$ for $x \leq 1$. Define $\overline{\theta}_i\colon M \to \mathbf{R}$ by $\overline{\theta}_i(p) := f(\|\phi_i(p)\|)$ if $p \in U_i$ and $\overline{\theta}_i(p) := 0$ otherwise. Then $\overline{\theta}_i$ are smooth. Finally, $\theta_i(p) := \overline{\theta}_i(p)/\sum_j \overline{\theta}_j(p)$, is the desired partition of unity. □

Recall that earlier we showed that any *compact* manifold admits a Riemannian metric, since it can be isometrically embedded in some Euclidean space. As an application of the previous result we now ca n show:

Corollary 0.8. *Any manifold admits a Riemannian metric*

Proof. Let (U_i, ϕ_i) be an atlas of M, and let θ_i be a subordinate partition of unity. Now for $p{\in}U_i$ define $g^i_p(X,Y) := \langle d\phi_i(X), d\phi_i(Y)\rangle$. Then we define a Riemannian metric g on M by setting $g_p(X,Y) := \sum_i \theta_i(p) g^i_p(X,Y)$. □

Exercise 0.9. Show that every manifold is *normal*, i.e., for every disjoint closed sets A_1, A_2 in M there exists a pair of disjoint open subsets U_1, U_2 of M such that $X_1 \subset U_1$ and $X_2 \subset U_2$. [*Hint:* Use the fact that every manifold admits a metric]

Exercise 0.10. Show that if U is any open subset of a manifold M and $A \subset U$ is a closed subset, then there exists smooth function $f \colon M \to \mathbf{R}$ such that $f = 1$ on A and $f = 0$ on $M - U$.

Exercise 0.11. Compute the volume (area) of a torus of revolution in $\mathbf{R}^3$.

Exercise 0.12. Let $M \subset \mathbf{R}^n$ be an embedded submanifold which may be parameterized by $f \colon U \to \mathbf{R}^n$, for some open set $U \subset \mathbf{R}^m$, i.e., f is a one-to-one smooth immersion and $f(U) = M$. Show that then $\mathrm{vol}(M) = \int_U \sqrt{\det(J_x(f) \cdot J_x(f)^T)}\, dx$, where $J_x(f)$ is the jacobian matrix of f at x.