

## Lecture Notes 13

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### 2.9 The Covariant Derivative, Lie Bracket, and Riemann Curvature Tensor of $\mathbf{R}^n$

Let  $A \subset \mathbf{R}^n$ ,  $p \in A$ , and  $W$  be a *tangent vector* of  $A$  at  $p$ , i.e., suppose there exists a curve  $\gamma: (-\epsilon, \epsilon) \rightarrow A$  with  $\gamma(0) = p$  and  $\gamma'(0) = W$ . Then if  $f: A \rightarrow \mathbf{R}$  is a function we define the (directional) derivative of  $f$  with respect to  $W$  at  $p$  as

$$W_p f := (f \circ \gamma)'(0) = df_p(W).$$

Similarly, if  $V$  is a *vectorfield* along  $A$ , i.e., a mapping  $V: A \rightarrow \mathbf{R}^n$ ,  $p \mapsto V_p$ , we define the *covariant derivative* of  $V$  with respect to  $W$  at  $p$  as

$$\bar{\nabla}_{W_p} V := (V \circ \gamma)'(0) = dV_p(W).$$

Note that if  $f$  and  $V$  are  $C^1$ , then by definition they may be extended to an open neighborhood of  $A$ . So  $df_p$  and  $dV_p$ , and consequently  $W_p f$  and  $\bar{\nabla}_{W_p} V$  are well defined. In particular, they do not depend on the choice of the curve  $\gamma$  or the extensions of  $f$  and  $V$ .

**Exercise 1.** Let  $E_i$  be the standard basis of  $\mathbf{R}^n$ , i.e.,  $E_1 := (1, 0, \dots, 0)$ ,  $E_2 := (0, 1, 0, \dots, 0), \dots, E_n := (0, \dots, 0, 1)$ . Show that for any functions  $f: \mathbf{R}^n \rightarrow \mathbf{R}$  and vectorfield  $V: \mathbf{R}^n \rightarrow \mathbf{R}^n$

$$(E_i)_p f = D_i f(p) \quad \text{and} \quad \bar{\nabla}_{(E_i)_p} V = D_i V(p)$$

(*Hint:* Let  $u_i: (-\epsilon, \epsilon) \rightarrow \mathbf{R}^n$  be given by  $u_i(t) := p + tE_i$ , and observe that  $(E_i)_p f = (f \circ u_i)'(0)$ ,  $\bar{\nabla}_{(E_i)_p} V = (V \circ u_i)'(0)$ ).

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The operation  $\bar{\nabla}$  is also known as the standard *Levi-Civita* connection of  $\mathbf{R}^n$ . If  $W$  is a tangent vectorfield of  $A$ , i.e., a mapping  $W: A \rightarrow \mathbf{R}^n$  such that  $W_p$  is a tangent vector of  $A$  for all  $p \in A$ , then we set

$$Wf(p) := W_p f \quad \text{and} \quad (\bar{\nabla}_W V)_p := \bar{\nabla}_{W_p} V.$$

Note that  $Wf: A \rightarrow \mathbf{R}$  is a function and  $\bar{\nabla}_W V$  is a vectorfield. Further, we define

$$(fW)_p := f(p)W_p.$$

Thus  $fW: A \rightarrow \mathbf{R}^n$  is also a vector field.

**Exercise 2.** Show that if  $V = (V^1, \dots, V^n)$ , i.e.,  $V^i$  are the component functions of  $V$ , then

$$\bar{\nabla}_W V = (WV^1, \dots, WV^n).$$

**Exercise 3.** Show that if  $Z$  is a tangent vectorfield of  $A$  and  $f: A \rightarrow \mathbf{R}$  is a function, then

$$\bar{\nabla}_{W+Z} V = \bar{\nabla}_W V + \bar{\nabla}_Z V, \quad \text{and} \quad \bar{\nabla}_{fW} V = f \bar{\nabla}_W V.$$

Further if  $Z: A \rightarrow \mathbf{R}^n$  is any vectorfield, then

$$\bar{\nabla}_W(V + Z) = \bar{\nabla}_W V + \bar{\nabla}_W Z, \quad \text{and} \quad \bar{\nabla}_W(fV) = (Wf)V + f \bar{\nabla}_W V.$$

**Exercise 4.** Note that if  $V$  and  $W$  are a pair of vectorfields on  $A$  then  $\langle V, W \rangle: A \rightarrow \mathbf{R}$  defined by  $\langle V, W \rangle_p := \langle V_p, W_p \rangle$  is a function on  $A$ , and show that

$$Z \langle V, W \rangle = \langle \bar{\nabla}_Z V, W \rangle + \langle V, \bar{\nabla}_Z W \rangle.$$

If  $V, W: A \rightarrow \mathbf{R}^n$  are a pair of vector fields, then their *Lie bracket* is the vector field on  $A$  defined by

$$[V, W]_p := \bar{\nabla}_{V_p} W - \bar{\nabla}_{W_p} V.$$

**Exercise 5.** Show that if  $A \subset \mathbf{R}^n$  is open,  $V, W: A \rightarrow \mathbf{R}^n$  are a pair of vector fields and  $f: A \rightarrow \mathbf{R}$  is a scalar, then

$$[V, W]f = V(Wf) - W(Vf).$$

(*Hint:* First show that  $Vf = \langle V, \text{grad } f \rangle$  and  $Wf = \langle W, \text{grad } f \rangle$  where

$$\text{grad } f := (D_1f, \dots, D_nf).$$

Next define

$$\text{Hess } f(V, W) := \langle V, \nabla_W \text{grad } f \rangle,$$

and show that  $\text{Hess } f(V, W) = \text{Hess } f(W, V)$ . In particular, it is enough to show that  $\text{Hess } f(E_i, E_j) = D_{ij}f$ , where  $\{E_1, \dots, E_n\}$  is the standard basis for  $\mathbf{R}^n$ . Then Leibnitz rule yields that

$$\begin{aligned} & V(Wf) - W(Vf) \\ &= V\langle W, \text{grad } f \rangle - W\langle V, \text{grad } f \rangle \\ &= \langle \nabla_V W, \text{grad } f \rangle + \langle W, \nabla_V \text{grad } f \rangle - \langle \nabla_W V, \text{grad } f \rangle - \langle V, \nabla_W \text{grad } f \rangle \\ &= \langle [V, W], \text{grad } f \rangle + \text{Hess } f(W, V) - \text{Hess } f(V, W) \\ &= [V, W]f, \end{aligned}$$

as desired.)

If  $V$  and  $W$  are tangent vectorfields on an open set  $A \subset \mathbf{R}^n$ , and  $Z: A \rightarrow \mathbf{R}^n$  is any vectorfield, then

$$\bar{R}(V, W)Z := \bar{\nabla}_V \bar{\nabla}_W Z - \bar{\nabla}_W \bar{\nabla}_V Z - \bar{\nabla}_{[V, W]}Z$$

defines a vectorfield on  $A$ . If  $Y$  is another vectorfield on  $A$ , then we may also define an associated scalar quantity by

$$\bar{R}(V, W, Z, Y) := \langle \bar{R}(V, W)Z, Y \rangle,$$

which is known as the *Riemann curvature tensor* of  $\mathbf{R}^n$ .

**Exercise 6.** Show that  $\bar{R} \equiv 0$ .

## 2.10 The Induced Covariant Derivative on Surfaces; Gauss's Formulas revisited

Note that if  $M \subset \mathbf{R}^3$  is a regular embedded surface and  $V, W: M \rightarrow \mathbf{R}^3$  are vectorfields on  $M$ . Then  $\bar{\nabla}_W V$  may no longer be tangent to  $M$ . Rather, in general we have

$$\bar{\nabla}_W V = (\bar{\nabla}_W V)^\top + (\bar{\nabla}_W V)^\perp,$$

where  $(\bar{\nabla}_W V)^\top$  and  $(\bar{\nabla}_W V)^\perp$  respectively denote the tangential and normal components of  $\bar{\nabla}_W V$  with respect to  $M$ . More explicitly, if for each  $p \in M$  we let  $n(p)$  be a unit normal vector to  $T_p M$ , then

$$(\bar{\nabla}_W V)_p^\perp := \langle \bar{\nabla}_{W_p} V, n(p) \rangle n(p) \quad \text{and} \quad (\bar{\nabla}_W V)^\top := \bar{\nabla}_W V - (\bar{\nabla}_W V)^\perp.$$

Let  $\mathcal{X}(M)$  denote the space of tangent vectorfield on  $M$ . Then We define the (*induced*) *covariant derivative* on  $M$  as the mapping  $\nabla: \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M)$  given by

$$\nabla_W V := (\bar{\nabla}_W V)^\top.$$

**Exercise 7.** Show that, with respect to tangent vectorfields on  $M$ ,  $\nabla$  satisfies all the properties which were listed for  $\bar{\nabla}$  in Exercises ?? and ??.

Next we derive an explicit expression for  $\nabla$  in terms of local coordinates. Let  $X: U \rightarrow M$  be a proper regular patch centered at a point  $p \in M$ , i.e.,  $X(0,0) = p$ , and set

$$\bar{X}_i := X_i \circ X^{-1}.$$

Then  $\bar{X}_i$  are vectorfields on  $X(U)$ , and for each  $q \in X(U)$ ,  $(\bar{X}_i)_q$  forms a basis for  $T_q M$ . Thus on  $X(U)$  we have

$$V = \sum_i V^i \bar{X}_i, \quad \text{and} \quad W = \sum_i W^i \bar{X}_i$$

for some functions  $V^i, W^i: X(U) \rightarrow \mathbf{R}$ . Consequently, on  $X(U)$ ,

$$\begin{aligned} \nabla_W V &= \nabla_{(\sum_j W^j \bar{X}_j)} \left( \sum_i V^i \bar{X}_i \right) \\ &= \sum_j \left( W^j \nabla_{\bar{X}_j} \left( \sum_i V^i \bar{X}_i \right) \right) \\ &= \sum_j \left( W^j \sum_i \left( \bar{X}_j V^i + V^i \nabla_{\bar{X}_j} \bar{X}_i \right) \right) \\ &= \sum_j \sum_i \left( W^j (\bar{X}_j V^i) + W^j V^i \nabla_{\bar{X}_j} \bar{X}_i \right). \end{aligned}$$

Next note that if we define  $u_j: (-\epsilon, \epsilon) \rightarrow \mathbf{R}^2$  by  $u_j(t) := tE_j$ , where  $E_1 := (1,0)$  and  $E_2 := (0,1)$ . Then  $X \circ u_i: (-\epsilon, \epsilon) \rightarrow M$  are curves with  $X \circ u_i(0) =$

$p$  and  $(X \circ u_i)'(0) = X_i(0, 0) = \bar{X}_i(p)$ . Thus by the definitions of  $\nabla$  and  $\bar{\nabla}$  we have

$$\begin{aligned}\nabla_{(\bar{X}_j)_p} \bar{X}_i &= \left( \bar{\nabla}_{(\bar{X}_j)_p} \bar{X}_i \right)^\top \\ &= \left( (\bar{X}_i \circ (X \circ u_j))'(0) \right)^\top \\ &= \left( (X_i \circ u_j)'(0) \right)^\top\end{aligned}$$

Now note that, by the chain rule,

$$(X_i \circ u_j)'(0) = DX_i(u_j(0))Du_j(0) = X_{ij}(0, 0).$$

**Exercise 8.** Verify the last equality above.

Thus, by Gauss's formula,

$$\begin{aligned}\nabla_{(\bar{X}_j)_p} \bar{X}_i &= \left( X_{ij}(0, 0) \right)^\top \\ &= \left( \sum_k \Gamma_{ij}^k(0, 0) X_k(0, 0) + l_{ij}(0, 0) N(0, 0) \right)^\top \\ &= \sum_k \Gamma_{ij}^k(X^{-1}(p)) X_k(X^{-1}(p)) \\ &= \sum_k \Gamma_{ij}^k(X^{-1}(p)) (\bar{X}_k)_p.\end{aligned}$$

In particular if we set  $\bar{X}_{ij} := X_{ij} \circ X^{-1}$  and define  $\bar{\Gamma}_{ij}^k: X(U) \rightarrow \mathbf{R}$  by  $\bar{\Gamma}_{ij}^k := \Gamma_{ij}^k \circ X^{-1}$ , then we have

$$\nabla_{\bar{X}_j} \bar{X}_i = (\bar{X}_{ij})^\top = \sum_k \bar{\Gamma}_{ij}^k \bar{X}_k,$$

which in turn yields

$$\nabla_W V = \sum_j \sum_i \left( W^j \bar{X}_j V^i + W^j V^i \sum_k \bar{\Gamma}_{ij}^k \bar{X}_k \right).$$

Now recall that  $\Gamma_{ij}^k$  depends only on the coefficients of the first fundamental form  $g_{ij}$ . Thus it follows that  $\nabla$  is intrinsic:

**Exercise 9.** Show that if  $f: M \rightarrow \widetilde{M}$  is an isometry, then

$$\widetilde{\nabla}_{df(W)}df(V) = df(\nabla_W V),$$

where  $\widetilde{\nabla}$  denotes the covariant derivative on  $\widetilde{M}$  (*Hint:* It is enough to show that  $\langle \widetilde{\nabla}_{df(\overline{X}_i)}df(\overline{X}_j), df(\overline{X}_l) \rangle = \langle df(\nabla_{\overline{X}_i}\overline{X}_j), df(\overline{X}_l) \rangle$ ).

Next note that if  $n: X(U) \rightarrow \mathbf{S}^2$  is a local Gauss map then

$$\langle \nabla_W V, n \rangle = -\langle V, \nabla_W n \rangle = -\langle V, dn(W) \rangle = \langle V, S(W) \rangle,$$

where, recall that,  $S$  is the shape operator of  $M$ . Thus

$$(\overline{\nabla}_{W_p} V)^\perp = \langle V, S(W_p) \rangle n(p),$$

which in turn yields

$$\overline{\nabla}_W V = \nabla_W V + \langle V, S(W) \rangle n.$$

This is Gauss's formula and implies the expression that we had derived earlier in local coordinates.

**Exercise 10.** Verify the last sentence.