Math 6455 Nov 22, 2006[1]
Differential Geometry I
Fall 2006, Georgia Tech

Lecture Notes 16

Exponential Map

0.1 ODE's revisited; Local flows of vector fields

Recall that earlier we proved:

Theorem 0.1. *Let $U \subset \mathbf{R}^n$ be an open set and $F\colon U \to \mathbf{R}^n$ be C^1, then for every $p_0 \in U$, there exists an $\overline{\epsilon} > 0$ such that for every $0 < \epsilon < \overline{\epsilon}$ there exists a* unique *curve $\alpha\colon (-\epsilon, \epsilon) \to U$ with $\alpha(0) = p_0$ and $\alpha'(t) = F(\alpha(t))$.*

Further recall that in the proof of the above theorem we showed that we may set

$$\overline{\epsilon} := \min\left\{\frac{r}{\sup_{\overline{B}_r(p_0)} \|F\|}, \frac{1}{\sqrt{n}\sup_{\overline{B}_r(p_0)} |D_j F^i|}\right\},$$

where r is any number which is chosen so small that $\overline{B}_r(p_0) \subset U$. Note that $\overline{\epsilon}$ depends continuously on r and p_0. Now let V be an open neighborhood of p_0 such that $\overline{V} \subset U$ and $\overline{B}_r(p) \subset U$ for all $p \in V$ and some fixed $r > 0$. Define $f\colon V \to \mathbf{R}$ by

$$f(p) := \min\left\{\frac{r}{\sup_{\overline{B}_r(p)} \|F\|}, \frac{1}{\sqrt{n}\sup_{\overline{B}_r(p)} |D_j F^i|}\right\}.$$

Then f is continuous and positive. Thus

$$\overline{\epsilon} := \inf_{\overline{V}} f > 0.$$

This shows that the above theorem may be restated in somewhat more general terms:

Theorem 0.2. *Let $U \subset \mathbf{R}^n$ be an open set and $F\colon U \to \mathbf{R}^n$ be C^1, then for every $p_0 \in U$, there is an open neighborhood $V \subset U$, $p_0 \in V$, and an $\overline{\epsilon} > 0$ such that for every $p \in V$ and $0 < \epsilon < \overline{\epsilon}$ there exists a* unique *curve $\alpha_p\colon (-\epsilon, \epsilon) \to U$ with $\alpha_p(0) = p$ and $\alpha_p'(t) = F(\alpha_p(t))$.* □

The above theorem allows us to define, for every $p_0 \in U$, a mapping $\alpha\colon (-\overline{\epsilon}, \overline{\epsilon}) \times V \to U$ by

$$\alpha(t, p) := \alpha_p(t)$$

[1]Last revised: December 6, 2006

where V is some open neighborhood of p_0. This mapping is called a *local flow* of the vector field F at p_0. The previous theorem states then that C^1 vector fields have a local flow at each point. Then next result shows that this flow is continuous:

Theorem 0.3. *Let $U \subset \mathbf{R}^n$ be open and $F: U \to \mathbf{R}^n$ be a C^1 vector field. Then, for every $p_0 \in U$ there exists an open neighborhood $V \subset U$, $p_0 \in V$, an $\overline{\epsilon} > 0$, and a* continuous *map $\alpha: (-\overline{\epsilon}, \overline{\epsilon}) \times V \to U$ which is a local flow of F.*

Proof. Let $(t_i, p_i) \in (-\overline{\epsilon}, \overline{\epsilon}) \times V$ be a sequence of points which converge to $(t, p) \in (-\overline{\epsilon}, \overline{\epsilon}) \times V$. Then

$$\|\alpha(p_i, t_i) - \alpha(p, t)\| = \|\alpha_{p_i}(t) - \alpha_p(t)\| \leq \delta(\alpha_{p_i}, \alpha_p)$$

where recall that

$$\delta(\alpha_p, \alpha_q) := \sup_{-\overline{\epsilon} \leq t \leq \overline{\epsilon}} \|\alpha_p(t) - \alpha_q(t)\|.$$

Thus it suffices to show that $\delta(\alpha_{p_i}, \alpha_p) \to 0$ as $p_i \to p$. To see this, first recall that, as we showed in the proof of Theorem 0.1, $\alpha_p(-\overline{\epsilon}, \overline{\epsilon}) \subset \overline{B}_r(p)$, where, as we discussed above, r is some constant such that $\overline{B}_r(p) \subset U$ for all $p \in V$. Further recall that if we let $C_p := C((-\overline{\epsilon}, \overline{\epsilon}), \overline{B}_r(p))$ denote the space of continuous curves from $(-\overline{\epsilon}, \overline{\epsilon})$ to $\overline{B}_r(p)$, then we have a mapping $s_p: c_p \to c_p$ given by

$$(s_p(\alpha))(t) = p + \int_0^t F(\alpha(t))dt,$$

which is a contraction with respect to δ, i.e.,

$$\delta(s_p(\alpha), s_p(\beta)) \leq K_p \delta(\alpha, \beta)$$

for some constant $K_p < 1$. Further note that

$$\delta(\alpha_p - S_q(\alpha_p)) = \delta(s_p(\alpha_p) - s_q(\alpha_p)) = \|p - q\|.$$

Thus, if s_q^n denotes the n^{th} iteration of s_q, it follows that

$$\begin{aligned}
\delta(\alpha_p, s_q^n(\alpha_p)) &\leq \delta(\alpha_p - s_q(\alpha_p)) + \delta(s_q(\alpha_p) - s_q^2(\alpha_p)) + \cdots + \delta(s_q^{n-1}(\alpha_p) - s_q^n(\alpha_p)) \\
&\leq (1 + K_q + \cdots + K_q^{n-1})\|p - q\| \\
&\leq \frac{1}{1 - K_q}\|p - q\|.
\end{aligned}$$

Now recall from the proof of the contraction mapping theorem for metric space that the fixed point α_q of s_q is the limit of $s_q^n(\beta)$ for any $\beta \in C_q$. In particular, $\lim_{n\to\infty} s_q^n(a_p) = \alpha_q$. Thus the last expression yields that

$$\delta(\alpha_p, \alpha_q) \leq \frac{1}{1 - K_q}\|p - q\|.$$

So we conclude that $\delta(\alpha_{p_i}, \alpha_p) \to 0$ as $p_i \to p$. □

One may also prove the following generalization of the above theorem.

Theorem 0.4. *The local flow* $\alpha\colon (-\bar{\epsilon}, \bar{\epsilon}) \times V \to U$ *is* C^1.

The proof of the above result is somewhat long and will be omitted for now.

0.2 Geodesic flow

Recall that if $\gamma\colon I \to M$ is a differentiable curve, then $\gamma'(t) = d\gamma_t(1) \in T_{\gamma(t)}M$ for every $t \in I$. Thus γ' may be considered the "natural lift" of γ from a curve on M to a curve on TM.

Lemma 0.5. *Let* M *be a Riemannian manifold with* C^2 *metric. Then there exists a* C^1 *vector field* F *on* TM *such* $\alpha\colon (-\epsilon, \epsilon) \to TM$ *is an integral curve of* F *if and only of* $\alpha = c'$ *for a geodesic* $c\colon (\epsilon, \epsilon) \to M$.

Proof. Let $v_0 \in TM$. Then $v_0 \in T_{p_0}M$ for some $p_0 \in M$. Recall that there exists a unique geodesic $c\colon (\epsilon, \epsilon) \to M$ with $c(0) = p_0$ and $c'(0) = v_0$. Then $\alpha := \gamma'$ is a curve on TM with $\alpha(0) = v_0$. Set $F(v_0) := \alpha'(0) = d\alpha_0(1)$. Then $F(v_0) \in T_{v_0}(T_pM)$. Thus we may define a vector field F on TM. To see that F is C^1 recall that if we identify a neighborhood of p_0 in M with $\mathbf{R}^n$ via some local charts, the a neighborhood of v_0 in TM may be identified with $\mathbf{R}^{2n}$ and F may be written as $F\colon \mathbf{R}^{2n} \to \mathbf{R}^{2n}$, $F = (F^1, \ldots, F^{2n})$, where

$$F^\ell(x, y) = y^\ell, \qquad \text{and} \qquad F^{\ell+n}(x, y) = -\sum_{ij} y^i y^j \Gamma^\ell_{ij}(x)$$

for $\ell = 1, \ldots, n$. Since Γ^ℓ_{ij} are obtained from the first derivatives of g_{ij} which are by assumption C^2, we conclude then that F is C^1. □

Lemma 0.6. *Let* $X \in T_pM$, $\alpha_X\colon (-\epsilon, \epsilon) \to M$ *be a geodesic with* $\alpha_X(0) = p$ *and* $\alpha'_X(0) = X$. *For* $\lambda > 0$, *define* $\alpha_{\lambda X}\colon (-\epsilon/\lambda, \epsilon/\lambda) \to M$ *by* $\alpha_{\lambda X}(t) := \alpha_X(\lambda t)$. *Then* $\alpha_{\lambda X}$ *is also a geodesic, and* $\alpha_{\lambda X}(0) = p$, $\alpha'_{\lambda X}(0) = \lambda X$.

Proof. First note that $\alpha_{\lambda X}$ is just a reparameterization of α_X. Secondly, by the chain rule, $\alpha'_{\lambda X}(t) = \lambda \alpha'_X(t)$. So $\alpha_{\lambda X}$ has constant speed, since α_X, being a geodesic, has constant speed. Finally recall that, as we showed earlier, any reparameterization of a geodesic is a geodesic if and only if it has constant speed. Thus $\alpha_{\lambda X}$ is a geodesic. □

Theorem 0.7. *Let* M *be a Riemannian manifold. For every* $p_0 \in M$ *an* $X_{p_0} \in T_{p_0}M$ *there exists an open neighborhood* U *of* X_{p_0} *in* TM, *and a* C^1 *mapping* $\alpha\colon (-2, 2) \times U \to M$ *such that for all* $X_p \in U$, $\alpha_{X_p}\colon (-2, 2) \to M$ *given by* $\alpha_{X_p}(t) := \alpha(t, X_p)$ *is a geodesic with* $\alpha_{X_p}(0) = p$ *and* $\alpha'_{X_p}(0) = X_p$.

Proof. Let F be as in the previous lemma. Then there exists a local flow $\phi\colon (-\delta,\delta)\times U \to TM$ of F. Let $\pi\colon TM \to M$ be the standard projection given by $\pi(T_pM) = p$ and define $\alpha(t, X_p) := \pi \circ \phi(\delta t/2, X_p)$. □

Corollary 0.8. *Let $B_\epsilon(p)$ be the set of vectors $X \in T_pM$ such that $\sqrt{g_p(X,X)} \le \epsilon$. For every $p \in M$ there exists an $\epsilon > 0$ such that for all $X \in B_\epsilon(p)$ there exists a geodesic $\alpha\colon (-2,2) \to M$ with $\alpha(0) = p$ and $\alpha'(0) = X$. Further the mapping $\exp_p\colon B_\epsilon(p) \to M$ defined by*

$$\exp_p(X) := \alpha_X(1)$$

is C^1.

Proof. By the previous theorem there exists an open neighborhood U of p in TM and a C^1 mapping $\alpha\colon (-2,2)\times U \to M$ such that $\alpha_{X_q}(t) := \alpha(t, X_q)$ is a geodesic for all $X_q \in U$. Let ϵ be so small that $B_\epsilon(p) \subset U$. Then α is well defined on $(-2,2)\times B_\epsilon(p)$ (and is still C^1). Consequently $\exp_p$ is C^1. □

The mapping defined above is called the *exponential map.*

Theorem 0.9. *The exponential map is a local diffeomorphism.*

Proof. By the inverse function theorem, it suffices to show that the differential $d(exp_p)_0\colon T_0(T_pM) \to T_pM$ is nonsingular. To see this first recall that if $f\colon M \to N$ is any function, then

$$df_p(X) = (f \circ \gamma)'(0)$$

where $\gamma\colon (-\epsilon,\epsilon) \to M$ is a curve with $\gamma(0) = p$ and $\gamma'(0) = d\gamma_0(1) = X$. Now note that if $X \in T_0(T_pM)$, then $X = [t\overline{X}]$ for some $\overline{X} \in T_pM$. Thus

$$d(exp_p)_0(X) = \frac{d}{dt}(exp(t\overline{X}))\Big|_{t=0}.$$

But

$$exp(t\overline{X}) = \alpha_{t\overline{X}}(1) = \alpha_{\overline{X}}(t).$$

So we conclude that

$$d(exp_p)_0(X) = \overline{X}.$$

In particular, if $X \neq 0$, then $d(exp_p)_0(X) \neq 0$ either. □