Math 426 Feb 26, 2004[1]
Introduction to Modern Geometry
Spring 2004, PSU

Lecture Notes 8

2 Surfaces

2.1 Definition of a regular embedded surface

An n-dimensional open ball of radius r centered at p is defined by

$$B^n_r(p) := \{\, x \in \mathbf{R}^n \mid \mathrm{dist}(x,p) < r \,\}.$$

We say a subset $U \subset \mathbf{R}^n$ is *open* if for each p in U there exists an $\epsilon > 0$ such that $B^n_\epsilon(p) \subset U$. Let $A \subset \mathbf{R}^n$ be an arbitrary subset, and $U \subset A$. We say that U is open in A if there exists an open set $V \subset \mathbf{R}^n$ such that $U = A \cap V$. A mapping $f\colon A \to B$ between arbitrary subsets of $\mathbf{R}^n$ is said to be *continuous* if for every open set $U \subset B$, $f^{-1}(U)$ is open in A. Intuitively, we may think of a continuous map as one which sends nearby points to nearby points:

Exercise 1. Let A, $B \subset \mathbf{R}^n$ be arbitrary subsets, $f\colon A \to B$ be a continuous map, and $p \in A$. Show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $\mathrm{dist}(x,p) < \delta$, then $\mathrm{dist}(f(x), f(p)) < \epsilon$.

Two subsets A, $B \subset \mathbf{R}^n$ are said to be *homeomorphic*, or topologically equivalent, if there exists a mapping $f\colon A \to B$ such that f is one-to-one, onto, continuous, and has a continuous inverse. Such a mapping is called a *homeomorphism*. We say a subset $M \subset \mathbf{R}^3$ is an *embedded surface* if every point in M has an open neighborhood in M which is homeomorphic to an open subset of $\mathbf{R}^2$.

Exercise 2. (Stereographic projection) Show that the standard sphere $\mathbf{S}^2 := \{p \in \mathbf{R}^3 \mid \|p\| = 1\}$ is an embedded surface (*Hint*: Show that the stereographic projection π_+ form the north pole gives a homeomorphism between $\mathbf{R}^2$ and $\mathbf{S}^2 - (0,0,1)$. Similarly, the stereographic projection π_-

[1] Last revised: October 15, 2021

from the south pole gives a homeomorphism between $\mathbf{R}^2$ and $\mathbf{S}^2 - (0,0,-1)$; $\pi_+(x,y,z) := (\frac{x}{1-z}, \frac{y}{1-z}, 0)$, and $\pi_-(x,y,z) := (\frac{x}{z-1}, \frac{y}{z-1}, 0)$).

Exercise 3. (Surfaces as graphs) Let $U \subset \mathbf{R}^2$ be an open subset and $f\colon U \to \mathbf{R}$ be a continuous map. Then

$$\operatorname{graph}(f) := \{(x, y, f(x,y)) \mid (x,y) \in U\}$$

is a surface. (*Hint*: Show that the orthogonal projection $\pi(x,y,z) := (x,y)$ gives the desired homeomorphism).

Note that by the above exercise the cone given by $z = \sqrt{x^2+y^2}$, and the troughlike surface $z = |x|$ are examples of embedded surfaces. These surfaces, however, are not "regular", as we will define below. From the point of view of differential geometry it is desirable that a surface be without sharp corners or vertices.

Let $U \subset \mathbf{R}^n$ be open, and $f\colon U \to \mathbf{R}^m$ be a map. Note that f may be regarded as a list of m functions of n variables: $f(p) = (f^1(p), \dots, f^m(p))$, $f^i(p) = f^i(p^1, \dots, p^n)$. The first order partial derivatives of f are given by

$$D_j f^i(p) := \lim_{h\to 0} \frac{f^i(p^1, \dots, p^j + h, \dots, p^n) - f^i(p^1, \dots, p^j, \dots, p^n)}{h}.$$

If all the functions $D_j f^i\colon U \to \mathbf{R}$ exist and are continuous, then we say that f is differentiable (C^1). We say that f is smooth (C^∞) if the partial derivatives of f of all order exist and are continuous. These are defined by

$$Dj_1, j_2, \dots, j_k f^i := D_{j_1}(D_{j_2}(\cdots(D_{j_k} f^i)\cdots)).$$

Let $f\colon U \subset \mathbf{R}^n \to \mathbf{R}^m$ be a differentiable map, and $p \in U$. Then the Jacobian of f at p is an $m \times n$ matrix defined by

$$J_p(f) := \begin{pmatrix} D_1 f^1(p) & \cdots & D_n f^1(p) \\ \vdots & & \vdots \\ D_1 f^m(p) & \cdots & D_n f^m(p) \end{pmatrix}.$$

We say that p is a *regular point* of f if the rank of $J_p(f)$ is equal to n. If f is regular at all points $p \in U$, then we say that f is regular.

Exercise 4 (Monge Patch)**.** Let $f\colon U \subset \mathbf{R}^2 \to \mathbf{R}$ be a differentiable map. Show that the mapping $X\colon U \to \mathbf{R}^3$, defined by $X(u^1, u^2) := (u^1, u^2, f(u^1, u^2))$ is regular (the pair (X, U) is called a *Monge Patch*).

If f is a differentiable function, then we define,

$$D_i f(p) := (D_i f^1(p), \dots D_i f^n(p)).$$

Exercise 5. Show that $f\colon U \subset \mathbf{R}^2 \to \mathbf{R}^3$ is regular at p if and only if

$$\|D_1 f(p) \times D_2 f(p)\| \neq 0.$$

Let $f\colon U \subset \mathbf{R}^n \to \mathbf{R}^m$ be a differentiable map and $p \in U$. Then the *differential* of f at p is a mapping from $\mathbf{R}^n$ to $\mathbf{R}^m$ defined by

$$df_p(x) := \lim_{t \to 0} \frac{f(p+tx) - f(p)}{t}.$$

Exercise 6. Show that (i)

$$df_p(x) = J_p(f) \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}.$$

Conclude then that (ii) df_p is a linear map, and (iii) p is a regular value of f if and only if df_p is one-to-one. Further, (iv) show that if f is a linear map, then $df_p(x) = f(x)$, and (v) $J_p(f)$ coincides with the matrix representation of f with respect to the standard basis.

By a *regular patch* we mean a pair (U, X) where $U \subset \mathbf{R}^2$ is open and $X\colon U \to \mathbf{R}^3$ is a one-to-one, smooth, and regular mapping. Furthermore, we say that the patch is *proper* if X^{-1} is continuous. We say a subset $M \subset \mathbf{R}^3$ is a *regular embedded surface*, if for each point $p \in M$ there exists a proper regular patch (U, X) and an open set $V \subset \mathbf{R}^3$ such that $X(U) = M \cap V$. The pair (U, X) is called a *local parameterization* for M at p.

Exercise 7. Let $f\colon U \subset \mathbf{R}^2 \to \mathbf{R}$ be a smooth map. Show that graph(f) is a regular embedded surface, see Exercise 4.

Exercise 8. Show that $\mathbf{S}^2$ is a regular embedded surface (*Hint:* (Method 1) Let $p \in \mathbf{S}^2$. Then p^1, p^2, and p^3 cannot vanish simultaneously. Suppose, for instance, that $p^3 \neq 0$. Then, we may set $U := \{u \in \mathbf{R}^2 \mid \|u\| < 1\}$, and let $X(u^1, u^2) := (u^1, u^2, \pm\sqrt{1 - (u^1)^2 - (u^2)^2})$ depending on whether p^3 is positive or negative. The other cases involving p^1 and p^2 may be treated similarly. (Method 2) Write the inverse of the stereographic projection, see Exercise 2, and show that it is a regular map).

The following exercise shows that smoothness of a patch is not sufficient to ensure that the corresponding surface is without singularities (sharp edges or corners). Thus the regularity condition imposed in the definition of a regular embedded surface is not superfluous.

Exercise 9. Let $M \subset \mathbf{R}^3$ be the graph of the function $f(x,y) = |x|$. Sketch this surface, and show that there exists a smooth one-to-one map $X \colon \mathbf{R}^2 \to \mathbf{R}^3$ such that $X(\mathbf{R}^2) = M$ (*Hint:* Let $X(x,y) := (e^{-1/x^2}, y, e^{-1/x^2})$, if $x > 0$; $X(x,y) := (-e^{-1/x^2}, y, e^{-1/x^2})$, if $X < 0$; and, $X(x,y) := (0,0,0)$, if $x = 0$).

The following exercise demonstrates the significance of the requirement in the definition of a regular embedded surface that X^{-1} be continuous.

Exercise 10. Let $U := \{(u,v) \in \mathbf{R}^2 \mid -\pi < u < \pi,\ 0 < v < 1\}$, define $X \colon U \to \mathbf{R}^3$ by $X(u,v) := (\sin(u), \sin(2u), v)$, and set $M := X(U)$. Sketch M and show that X is smooth, one-to-one, and regular, but X^{-1} is not continuous.

Exercise 11 (Surfaces of Revolution)**.** Let $\alpha \colon I \to \mathbf{R}^2$, $\alpha(t) = (x(t), y(t))$, be a regular simple closed curve. Show that the image of $X \colon I \times \mathbf{R} \to \mathbf{R}^3$ given by

$$X(t,\theta) := \Big(x(t)\cos\theta, x(t)\sin\theta, y(t)\Big),$$

is a regular embedded surface.