Preprint of a paper published in
Topology, **41** (2002) 107–117.

GAUSS MAP, TOPOLOGY, AND CONVEXITY OF HYPERSURFACES WITH NONVANISHING CURVATURE

MOHAMMAD GHOMI

ABSTRACT. It is proved that, for $n \geqslant 2$, every immersion of a compact connected n-manifold into a sphere of the same dimension is an embedding, if it is one-to-one on each boundary component of the manifold. Some applications of this result are discussed for studying geometry and topology of hypersurfaces with non-vanishing curvature in Euclidean space, via their Gauss map; particularly, in relation to a conjecture of Meeks on minimal surfaces with convex boundary. It is also proved, as another application, that a compact hypersurface with nonvanishing curvature is convex, if its boundary lies in a hyperplane.

1. INTRODUCTION

The purpose of this paper is twofold: first, to prove a basic fact in topology; and second, to develop some of its applications in differential geometry.

1.1. **A Topological Lemma.** Let X and Y be topological spaces. We say a mapping $f: X \to Y$ is an *immersion*, if it is continuous and locally one-to-one. f is said to be an *embedding*, if it is a homeomorphism onto its image; or, more explicitly, if it is continuous, one-to-one, and for every open subset $U \subset X$ there exists an open subset $V \subset Y$ such that $f(U) = V \cap f(X)$. The main lemma used in this paper is the following:

Lemma 1.1. *Let M be a compact connected n-manifold, $n \geqslant 2$, and $f: M \to \mathbf{S}^n$ be an immersion. Suppose that f is one-to-one on each boundary component of M. Then f is an embedding.*

Neither f nor M are required to be differentiable. By an *n-manifold M* we mean a Hausdorff topological space with a countable basis and the property that each point $p \in M$ has an open neighborhood which is homeomorphic either to the Euclidean space $\mathbf{R}^n$ or the upper half-space $\mathbf{H}^n$. If every neighborhood of p is homeomorphic to $\mathbf{H}^n$, we say that p is a boundary point, $p \in \partial M$.

Lemma 1.1 is proved in Section 2. The proof is based on Jordan-Brouwer separation theorem, and follows from certain gluing techniques which are elaborated in detail.

1991 *Mathematics Subject Classification.* Primary 57N35, 53A05; Secondary 53A10.

Key words and phrases. Manifold with boundary, collar, immersion, embedding, gauss map, mean curvature, convex cap.

First draft, January 1998; last typeset: September 9, 2002.

1.2. **Applications to Geometry.** In Sections 3, we will discuss some applications of Lemma 1.1 for studying hypersurfaces in Euclidean space via their Gauss map. The central observation utilized there, see Corollary 3.1, is that when the Gauss-Kronecker curvature of a surface does not vanish then its Gauss map gives an immersion into the sphere. Further, the requirement that the mapping be one-to-one on each boundary component is met within certain natural contexts, as discussed in Notes 3.2, and thus allows us to apply the lemma. The main applications are Theorems 3.3 and 3.5. The first theorem is concerned with the topology of minimal hypersurfaces, motivated by a well-known conjecture of W. Meeks, and the second theorem gives a local characterization for convex caps, inspired by Hadamard's classical theorem on ovaloids.

2. Proof of the Lemma

To prove Lemma 1.1, it suffices to show that f is one-to-one everywhere. This is due to the fact that any one-to-one continuous mapping $f\colon X \to Y$ is an embedding, if X is compact and Y is Hausdorff. The simple argument is worth recalling at this point. Let $U \subset X$ be an open subset. Then $X - U$, being a closed subset of a compact space, is compact. Therefore, $f(X - U)$, being a compact subset of a Hausdorff space, is closed. Let $V := Y - f(X - U)$. Then V is open. Furthermore, since f is one-to-one, it follows that $V \cap f(X) = f(U)$. Hence $f\colon X \to f(X)$ is a homeomorphism.

2.1. **Basic Strategy.** To show that f is one-to-one recall that, since $\mathbf{S}^n$ is simply connected (assuming $n \geqslant 2$), then every covering map of $\mathbf{S}^n$ by a connected space has to be one-to-one. Thus, all we need is to construct a pair $(\widetilde{M}, \widetilde{f})$ with the following properties: (i) $\widetilde{M}$ is connected, and admits an embedding $i\colon M \to \widetilde{M}$; (ii) $\widetilde{f}\colon \widetilde{M} \to \mathbf{S}^n$ is a covering map such that $\widetilde{f} \circ i = f$. In particular, we need to show that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{i} & \widetilde{M} \\ & \searrow f & \big\downarrow \widetilde{f} \\ & & \mathbf{S}^n \end{array}$$

This line of approach was first suggested to the author by Herman Gluck [1]. We should point out that this method can be carried out not only in the smooth category, where f is necessarily well behaved, but also in the topological case, where f may be some kind of a *wild embedding* such as the famous Alexander's horned sphere, or other examples due to Fox and Artin [2]. The proof is organized into two parts: first we construct $\widetilde{M}$, and then $\widetilde{f}$.

2.2. **Construction of $\widetilde{M}$.** Let Γ_i, $1 \leqslant i \leqslant N$, be a component of ∂M (since M is compact, there are only finitely many components), and let U_i be a *collar* of Γ_i, i.e., suppose there exists an embedding $c_i\colon \Gamma_i \times [0,1) \to M$ such that $c_i(x,0) = x$ for all

$x \in \Gamma_i$, and $c_i(\Gamma_i \times [0,1)) = U_i$[1]. In 2.2.1 below we will prove that there exists a collar U_i of Γ_i such that $f\big|_{U_i}$ is one-to-one[2]. This shows that $f(U_i - \Gamma_i) \subset \mathbf{S}^n - f(\Gamma_i)$. Now note that since Γ_i is a closed manifold of dimension $n-1$, and $\mathbf{S}^n$ is simply connected, then, by a well-known generalization of the Jordan-Brouwer separation theorem, $\mathbf{S}^n - f(\Gamma_i)$ has exactly two components, say D_i and D_i' [3]; therefore, $f(U_i - \Gamma_i)$ must lie entirely within either D_i or D_i', because $U_i - \Gamma_i = c_i(\Gamma_i \times (0,1))$ is connected. Let D_i' be the component which contains $f(U_i - \Gamma_i)$, see Figure 1.

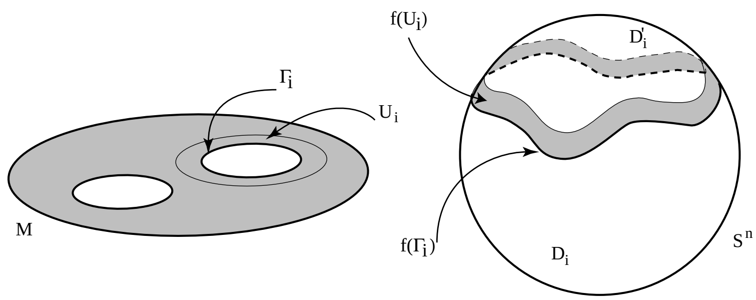

FIGURE 1

Next, note that the (topological) boundary of D_i (as a subset of $\mathbf{S}^n$), is $f(\Gamma_i)$. Denote this boundary by $\mathrm{bd}(D_i)$, and set $\overline{D}_i := D_i \cup \mathrm{bd}(D_i)$. Let $\partial f_i := f|_{\Gamma_i}$. Then $\partial f_i \colon \Gamma_i \to \mathrm{bd}(D_i)$ is a homeomorphism. Thus we can use each ∂f_i to *glue* $\overline{D_i}$ to M along the corresponding boundaries, see Figure 2. This means forming the

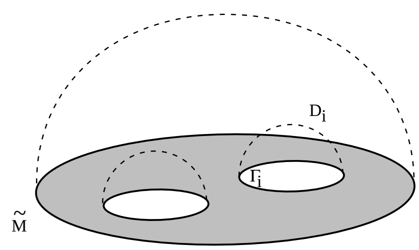

FIGURE 2

adjunction space

$$\widetilde{M} := (M \cup_{\partial f_1} \overline{D}_1) \cdots \cup_{\partial f_N} \overline{D}_N.$$

[1]The existence of collars for differentiable manifolds is well-known; it follows from the tubular neighborhood theorem. Since we do not make any differentiability assumptions, however, we refer to a result of M. Brown [3] who proved the existence of collars for topological manifolds. See Also the paper by R. Connelly [4].

[2]This does not preclude the possibility for a wild embedding. In order for f to be *tame*, a somewhat stronger *bicollaring* property, as studied by B. Mazur [5], is needed. See also related papers by M. Brown [6], and M. Morse [7].

[3]The proof of this fact in the topological category follows from Alexander duality, see [8, Pg. 353]; for the smooth case, there is a proof [9, Pg. 107] based on intersection theory.

Some basic facts which we need to know about this construction are as follows. First, the elements of $\widetilde{M}$ are the equivalence classes $[x]$, where $x \in M \cup_i \overline{D}_i$. These are defined by

$$[x] := \begin{cases} \{x, f(x)\}, & \text{if } x \in \partial M; \\ \{\partial f_i^{-1}(x), x\}, & \text{if } x \in \operatorname{bd} D_i; \\ \{x\}, & \text{otherwise.} \end{cases}$$

Secondly, the topology of $\widetilde{M}$ is generated by the *quotient map* $p\colon M \cup_i \overline{D}_i \to \widetilde{M}$, $p(x) := [x]$; it consists of all subsets $U \subset \widetilde{M}$ such that $p^{-1}(U)$ is open. In particular, note that a mapping $f\colon M \to X$ is continuous if $f \circ p$ is continuous. Also note that if $U \subset M \cup_i \overline{D}_i$ is a *saturated* subset, i.e., $[x] \in U$ whenever $x \in U$; then, $p^{-1}(p(U)) = U$. Hence p maps saturated open subsets to open subsets.

Next we verify a claim which was made earlier, i.e., that f is one-to-one on a collar of each boundary component of M; and then show that $\widetilde{M}$ has the desired properties, i.e., $\widetilde{M}$ is connected and admits an embedding of M.

2.2.1. *The injectivity of f on collars of M.* Let $c_i\colon \Gamma_i \times [0,1) \to M$ be a *collaring* of M around the boundary component Γ_i. For every $n \in \mathbb{N}$ set $U_{i,n} := c_i(\Gamma_i \times [0, 1/n))$. Then each $U_{i,n}$ is a collar of M around Γ_i. Furthermore, note that (i) $\overline{U}_{i,n+1} \subset U_{i,n}$, and (ii) $\cap_n U_{i,n} = \Gamma_i$. We are going to show that there exists an $n \in \mathbb{N}$ such that $f|_{U_{i,n}}$ is one-to-one. The proof is by contradiction; suppose that for every $n \in \mathbb{N}$ there exists a pair of points x_n, $y_n \in U_{i,n}$ such that $f(x_n) = f(y_n)$, but $x_n \neq y_n$. Since M is compact, each of these sequences must have a limit point, say x and y respectively. From (i) and (ii) it follows that x and $y \in \Gamma_i$. Also note that, since f is continuous, $f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n) = f(y)$. This implies that $x = y$, because $f|_{\Gamma_i}$ is one-to-one by assumption. Now we have a contradiction, because f has to be locally one-to-one in a neighborhood of the point $x = y$, but every such neighborhood contains a pair of points x_n and y_n for some $n \in \mathbb{N}$.

2.2.2. *Connectedness of $\widetilde{M}$.* Let $U \subset \widetilde{M}$ be open, closed, and nonempty. We are going to show that $U = \widetilde{M}$. To see this note that, since p is continuous, $p^{-1}(U)$ is open and closed in $M \cup_i \overline{D}_i$. Thus, if $p^{-1}(U)$ contains a point of M or a point of $\overline{D}_i$, then it must contain all of M or all of $\overline{D}_i$ respectively. Furthermore, note that $p^{-1}(U)$, being a saturated subset, contains a point of Γ_i, if, and only if, it contains a point of $\operatorname{bd}(D_i)$; therefore, $p^{-1}(U)$, being nonempty, must contain all of $M \cup_i \overline{D}_i$. Hence $U = p(p^{-1}(U)) = p(M \cup_i \overline{D}_i) = \widetilde{M}$.

2.2.3. *The natural embedding of M in $\widetilde{M}$.* Let $i\colon M \to \widetilde{M}$ be given by $i(x) := [x]$. i is continuous, because it is the restriction of p. Furthermore, i is one-to-one; because if $i(x) = i(y)$, then $[x] = [y]$ which implies $y \in [x]$. Since $x \in M$, either $[x] = \{x\}$ or $[x] = \{x, f(x)\}$. Therefore, we must have $y = x$, because y is a point of M and $f(x)$ is not.

2.3. **Construction of $\widetilde{f}$.** Define $\widetilde{f}\colon \widetilde{M} \to \mathbf{S}^n$ by

$$\widetilde{f}([x]) := \begin{cases} f(x), & \text{if } x \in M; \\ x, & \text{otherwise.} \end{cases}$$

It is not difficult to see that $\widetilde{f}$ is well defined. Also, it is immediate from the definitions that $\widetilde{f} \circ i = f$. Hence, all is left is proving that $\widetilde{f}$ is a covering map. To this end, since $\widetilde{M}$ is compact and $\mathbf{S}^n$ is connected, it suffices to show that $\widetilde{f}$ is a local homeomorphism. We prove this, via Brouwer's theorem on invariance, by showing that $\widetilde{f}$ is continuous, locally one-to-one, and $\widetilde{M}$ is an n-manifold without boundary.

2.3.1. *The continuity of $\widetilde{f}$.* By the definition of the quotient topology, it suffices to show that $\widetilde{f} \circ p$ is continuous. Let $\overline{f}\colon M \cup_i \overline{D}_i \to \mathbf{S}^n$ be given by $\overline{f}(x) := f(x)$, if $x \in M$; and, $\overline{f}(x) := x$, otherwise. Then $\widetilde{f} \circ p = \overline{f}$. Furthermore, $\overline{f}$, being the union of continuous functions on disjoint spaces is continuous.

2.3.2. *The local injectivity of $\widetilde{f}$.* Let $[x] \in \widetilde{M}$. We are going to show that that there exists an open subset $U \subset \widetilde{M}$ such that $[x] \in U$, and $\widetilde{f}|_U$ is one-to-one. There are three cases to consider: (a) $x \in D_i$, (b) $x \in \mathrm{int}(M)$, and (c) $x \in \Gamma_i$ or $x \in \mathrm{bd}(D_i)$.

(a)Let $U = p(D_i)$, then U is open, because D_i is a saturated open subset. Suppose $\widetilde{f}([x]) = \widetilde{f}([y])$ for some $[x]$, $[y] \in U$, then x, $y \in D_i$. Hence $\widetilde{f}([x]) = x$, and $\widetilde{f}([y]) = y$. Therefore $x = y$.

(b)Let $U = p(V)$, where V is an open subset of $\mathrm{int}(M)$ such that $x \in V$, and $f|_V$ is one-to-one. Recall that f is locally one-to-one by assumption, so V exists. Moreover V is saturated; therefore, U is open. Now suppose $\widetilde{f}([x]) = \widetilde{f}([y])$ for some $[x]$, $[y] \in U$, then x, $y \in V$. Hence $\widetilde{f}([x]) = f(x)$, and $\widetilde{f}([y]) = f(y)$. Therefore $f(x) = f(y)$, which yields $x = y$.

(c)Let $U = p(\overline{D}_i \cup U_i)$ where U_i is a collar of Γ_i such that $f|_{U_i}$ is one-to-one, and $f(U_i - \Gamma_i) \subset D_i'$. Then U, being the image of an open saturated subset, is open. Suppose $\widetilde{f}([x]) = \widetilde{f}([y])$ for some $[x]$, $[y] \in U$, then either (1) $x \in D_i$, (2) $x \in \mathrm{bd}(D_i)$, (3) $x \in \mathrm{int}(M)$, or (4) $x \in \Gamma_i$. Similarly y can belong to any of these four locations. So we have 16 combinations. Of these we need to consider only 8, due to symmetry. Furthermore, we have already considered two of these cases in parts (a) and (b). Verifying the rest is also straight forward, so we omit further details.

2.3.3. *$\widetilde{M}$ is an n-manifold without boundary.* If f is differentiable, or satisfies some weaker "niceness" condition such as *locally flat*, then $\overline{D}_i$ is a manifold with boundary $\partial\overline{D}_i = f(\Gamma_i)$. In this case, it is well known that $\widetilde{M}$ would be a manifold without boundary. For the general case, however, we need to do more work. Suppose $[x] \in \widetilde{M}$. If $x \in \mathrm{int}(M)$ or $x \in D_i$, then it is clear that $[x]$ has an open neighborhood U which is homeomorphic to $\mathbf{R}^n$. So suppose that $x \in \Gamma_i$ or $x \in \mathrm{bd}(D_i)$. In this case, as in part (c) of 2.3.2, let $U := p(\overline{D}_i \cup U_i)$. We claim that $\widetilde{f}(U)$ is open and $\widetilde{f}|_U\colon U \to f(U)$ is a homeomorphism. We have already established that $\widetilde{f}|_U$

is continuous and one-to-one (in 2.3.1 and 2.3.2 respectively). Furthermore, it is clear that $\widetilde{f}(U) = f(U_i) \cup \overline{D}_i$ is open. Finally, it is also easy to see that $\widetilde{f}^{-1}|_{\widetilde{f}(U)}$ is continuous, because $\widetilde{f}^{-1}|_{\overline{D}_i}$ and $\widetilde{f}^{-1}|_{f(U_i)}$ are continuous, and we can use the gluing lemma for continuous functions. This completes the proof that $\widetilde{M}$ is locally homeomorphic to $\mathbf{R}^n$. It can be shown also that $\widetilde{M}$ is Hausdorff and has a countable basis; however, we do not need these properties to prove that $\widetilde{f}$ is a covering map.

Remark 2.1. Since the essential property of the target space, $\mathbf{S}^n$, used in this section was its separation property in the sense of Jordan-Brouwer theorem, the proof given here for Lemma 1.1 works just as well if we replace $\mathbf{S}^n$ by any compact, connected, and simply connected n-manifold without boundary.

3. Applications

3.1. **Preliminaries.** If M is a smoothly immersed hypersurface in Euclidean Space $\mathbf{R}^{n+1}$, then throughout an open neighborhood U of each point $p \in M$, we can continuously assign a unit vector $\nu(q) \in \mathbf{S}^n$ which is normal to M at every $q \in U$. Further, since M is smooth, ν is smooth as well. In particular, the *differential map*, ν_*, is well defined. The determinant of this map is the *Gauss-Kronecker* curvature of M:

$$K(p) := \det(\nu_{*_p}).$$

If M is *orientable*, then the mapping $\nu\colon M \to \mathbf{S}^n$ is well defined globally and known as the *Gauss map* of M. An immediate consequence of Lemma 1.1 is the following:

Corollary 3.1. *Let $M \subset \mathbf{R}^n$ be a compact connected orientable hypersurface with non-vanishing Gauss-Kronecker curvature. Suppose the Gauss map of M is one-to-one on each boundary component of M. Then M is diffeomorphic to its spherical image.*

Proof. If $K \neq 0$, then, by the inverse function theorem, ν has to be a local diffeomorphism. In particular, ν is locally one-to-one. This, via Lemma 1.1, implies that ν is a diffeomorphism. □

Notes 3.2. The condition that the Gauss map be one-to-one on a boundary component Γ of a hypersurface M is not an unnatural one. It occurs, for instance, when M has positive curvature, and Γ lies embedded in a hyperplane, see [10]. More generally, whenever (i) Γ lies in a hyperplane H, (ii) Γ is *strictly convex*, i.e., Γ contains no line segments and lies on the boundary of a convex body $K \subset H$, and (iii) M meets H transversely, then the Gauss map is one-to-one on Γ. To see this, let $p \in \Gamma$, then $\nu(p)$ cannot be orthogonal to H. In particular, the projection of $\nu(p)$ into H, $\overline{\nu}(p)$, does not vanish, see Figure 3. Further, note that $\overline{\nu}(p)$ is normal to Γ in H; therefore, $\overline{\nu}$ has to one-to-one on Γ, because Γ is strictly convex. Hence ν has to be one-to-one on Γ as well.

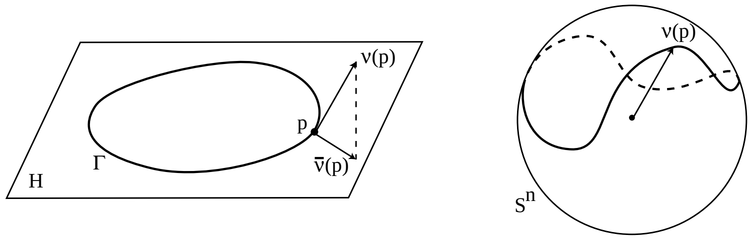

FIGURE 3

3.2. **Meeks's Conjecture.** Here we mention an application of Lemma 1.1 motivated by a well-known conjecture of W. Meeks [11, Conjecture 16]: "a compact connected minimal surface in $\mathbf{R}^3$ with boundary curves being two convex Jordan curves on parallel planes is topologically an annulus". A hypersurface $M \subset \mathbf{R}^{n+1}$ is said to be *minimal*, if its *mean curvature* vanishes. The mean curvature is determined by the trace of the differential of the Gauss map:

$$H(p) := \frac{1}{n}\,\mathrm{trace}(v_{*_p}).$$

Theorem 3.3. *Let $M \subset \mathbf{R}^{n+1}$ be an immersed compact connected orientable minimal hypersurface. Suppose that each boundary component of M lies in a hyperplane and is convex. Then M is diffeomorphic to its spherical image via the Gauss map, provided its Gauss-Kronecker curvature does not vanish. In particular, M is topologically an annulus if it has exactly two boundary components, and is two dimensional ($n = 2$).*

Proof. From Hopf's (boundary) maximum principle [12] it follows that M meets each hyperplane transversely along the corresponding boundary component, see [13, Lemma 1]. Further, since ∂M is compact it cannot contain any line segments (we are assuming that our minimal hypersurfaces are analytic up to the boundary). These considerations yield that the Gauss map is one-to-one on each component, see Remark 3.2. Hence, by Corollary 3.1, the Gauss map, ν, is a diffeomorphism because, by assumption, the Gauss-Kronecker curvature does not vanish (the Gauss map is globally well-defined due to the orientability assumption). If $n = 2$, then each component of ∂M is topologically a circle. Thus $\nu(\partial M)$ consists of a number of disjoint simple closed curves. If ∂M has only two components, then $\mathbf{S}^n - \partial M$ will have exactly three components: two disks and one annulus. But the annulus is the only component bounded by both components of ∂M. Thus $\nu(M)$ must be this annulus. □

For a nice introduction to Meeks's conjecture and further references see [14]. Some related results may be found in [13], [15], [16], and [17].

Remark 3.4. From Corollary 3.3 it follows that a counterexample to Meeks's conjecture, if it exists, must have points where the Gauss-Kronecker curvature vanishes. Further, note that when $n = 2$, then at each point $p \in M$ where $K(p) = 0$, both eigenvalues of v_{*_p} (the *principal curvatures*) have to vanish. In this case, p is called a *flat point*. It follows from the maximum principle that flat points of a nontrivial minimal surface have to be isolated. Thus there are only a finite number of such points if the surface is compact. It seems that a counterexample to Meeks's conjecture must have at least four flat points, See Figure 4. Since each flat point of

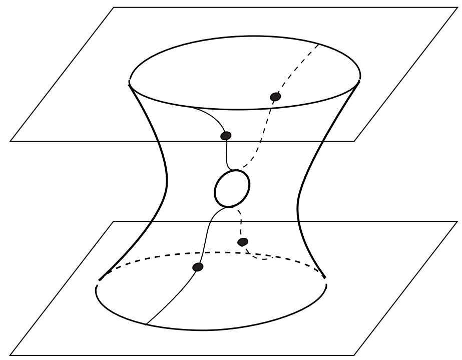

FIGURE 4

M corresponds to a *branch point* of ν, the *Riemann-Hurwitz formula* [18, Pg. 216] can be used to count the minimum number of flat points needed for constructing a counterexample with a given genus.

3.3. **Convex Caps.** We say a hypersurface $M \subset \mathbf{R}^{n+1}$ is *convex*, if it lies on the boundary of a convex body $K \subset \mathbf{R}^{n+1}$. Here we show:

Theorem 3.5. *Let $M \subset \mathbf{R}^{n+1}$ be an immersed compact connected hypersurface with non-vanishing Gauss-Kronecker curvature. Suppose that ∂M lies in a hyperplane H, and, furthermore, either $n > 2$ or each component of ∂M is embedded. Then M is convex. In particular, M is embedded and homeomorphic to a disk.*

Proof. It suffices to show that for every $p \in M$ the tangent plane T_pM supports M, i.e., M lies entirely in one of the closed half-spaces determined by T_pM.

First note that the Gauss map of M is globally well defined, for at the farthest point of M from H, the surface has to lie on one side of its tangent plane. This implies that the curvature has to be nonnegative at one point, which in turn implies that it has to be positive everywhere. In particular, the surface is locally strictly convex, i.e., each point $p \in M$ has a neighborhood U which lies strictly on one side of the tangent plane T_pM. Define the outward unit normal $\nu(p)$ to be the unit normal to M at p which points into the half-space, determined by T_pM, not containing U. Then it is easy to see that $\nu\colon M \to \mathbf{S}^n$ is continuous.

Next, note that ∂M is a positively curved hypersurface in its own hyperplane H. Thus, by a well-known theorem of Hadamard [19, Pg. 119], each component of M is strictly convex, provided that $n > 2$. Furthermore, if $n = 2$, then each boundary component is embedded by assumption. Therefore, by a well-known classical result in differential geometry [20, Pg. 21], each boundary component is again strictly convex. From this it follows that the restriction of ν to each boundary component of M is always one-to-one (see Remark 3.2). Hence ν is one-to-one everywhere, by Corollary 3.1.

Now observe that T_pM supports M if and only if T_pM supports ∂M. To see this let T_pM^+, and T_pM^- denote the closed half spaces determined by T_pM. Suppose that $\partial M \subset T_pM^+$. Further, suppose, towards a contradiction, that there exists a point of M in the interior of T_pM^-. Then there must be one such point, say q, which is farthest away from T_pM. Consequently, T_pM and T_qM will be parallel, and, since $\nu(p) \neq \nu(q)$, it follows that $\nu(p) = -\nu(q)$. We claim that this is impossible. To see this, suppose that H (the hyperplane containing ∂M) is given by the set of points in $\mathbf{R}^n$ whose n^{th} coordinate is zero. Then $\nu(\partial M)$ will lie in $\mathbf{S}^n$ with the North and South poles deleted, because M, having positive curvature, meets H transversely. Further, note that since each component of ∂M is strictly convex, then it is homeomorphic to $\mathbf{S}^{n-1}$. Furthermore, it is easy to see that the image of each component is homotopic to the equator in the complement of the poles: let Γ be a component of ∂M, let $\overline{\nu}$ denote the projection of ν into H (see Figure 3), and define $h\colon \Gamma \times [0,1] \to \mathbf{S}^n$ by

$$h(p,t) := \frac{(1-t)\nu(p) + t\overline{\nu}(p)}{\|(1-t)\nu(p) + t\overline{\nu}(p)\|}.$$

h is the desired homotopy. Hence, the image of each component of ∂M separates the North and South poles. Consequently, $\nu(M)$ cannot contain both poles, and we have our contradiction.

So it remains to show that $\partial M \subset T_pM^+$ for every $p \in M$ (by convention, we assume that T_pM^+ is the half space which contains a neighborhood of p, so that $v(p)$ points into T_pM^-). Let

$$X := \{p \in M \mid \partial M \subset T_pM^+\}.$$

Then $X \neq \emptyset$, for it has to contain a point of M which is at the farthest distance from H. Further, it is clear that X is closed, because the limit of supporting hyperplanes is a supporting hyperplane. Thus, since M is connected, it remains to show that X is open.

Let $p \in X$. If $T_pM \cap \partial M = \emptyset$, then it is clear that X contains an open neighborhood of p. So suppose that T_pM contains a point x of ∂M. Note that T_pM cannot coincide with H. To see this, let H^+ and H^- denote the half-spaces determined by H. If M is tangent to H at a point p, then $\nu(p)$ has to point either into H^+ or H^-. Suppose that ν points into H^-. Then a neighborhood U of p has to lie in H^+. In particular, there will be interior points of M in the interior of H^-. At a farthest such point, say q, we have to have $\nu(q) = -\nu(p)$, which is impossible as was explained before. Consequently, T_pM and H will have to meet along an $n-1$-dimensional subspace l, see Figure 5.

We claim that T_pM is tangent to M at x. Let U be a small neighborhood of M at x. Then U has to lie on one side of H, say H^+. Further, since T_pM supports M, $U \subset T_pM^+$. Thus U lies inside a wedge-shaped region $W := H^+ \cap T_pM^+$. This shows that T_xM cannot meet W only along l. Further, T_xM cannot coincide with H because U meets H transversely. Furthermore, T_xM cannot pass through

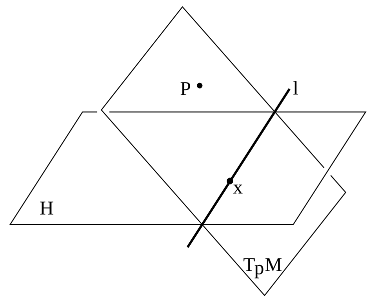

FIGURE 5

the interior of W, for any such plane would separate p and ∂M (recall that any supporting plane for ∂M has to support M). So the only possibility is that T_xM coincides with T_pM. This implies that $\nu(x) = \nu(p)$, which in turn yields $x = p$.

So T_pM meets ∂M at exactly one point: p. Further, note that T_pM cannot meet any interior point of M, for that would violate the injectivity of ν. Thus $T_pM \cap M = \{p\}$. This, together with the fact that M is locally strictly convex at p, easily yields that $T_qM \cap M = \{q\}$ for all points q in a sufficiently small open neighborhood of p. Hence X is open. □

Note 3.6. Theorem 3.5, or at least some similar versions of it, may be proved using a number of other methods. One, see [21, Main Lemma], is based purely on local convexity and uses no smoothness assumptions. Another, [22, Thm. 4], uses Banchoff's two-piece-property, and assumes only that the sectional curvatures be nonnegative. Still another general proof follows from [23, Thm 1.2.4], see also [24], where the boundary is not restricted to lie in a hyperplane. However, the proof of Theorem 3.5 presented here is shorter and more direct. Finally, we should point out that the theorem remains true without the embeddedness assumption on the boundary components, but the proof requires more work. See [25] for a number of more related results and generalizations.

Acknowledgments. The author is indebted to Herman Gluck for a very helpful conversation with regard to the proof of Lemma 1.1, and to Bill Meeks for bringing his conjecture to the author's attention.

REFERENCES

[1] H. Gluck, Personal communication.
[2] T. Rushing, *Topological Embeddings*, Academic Press, New York, 1973.
[3] M. Brown, *Locally flat embeddings of topological manifolds*, Ann. of Math. **75**(1962), 59–65.
[4] R. Connelly, *A new proof of Brown's collaring theorem*, Proc. Amer. Math. Soc. **27**(1971), 180–182.
[5] B. Mazur, *On embeddings of spheres*, Bull. Amer. Math. Soc. **65**(1959), 59–65.
[6] M. Brown, *A proof of the generalized Schoenflies theorem*, Bull. Amer. Math. Soc. **66**(1960), 74–76.
[7] M. Morse, *A reduction of the Schoenflies extension problem*, Bull. Amer. Math. Soc. **66**(1960), 113–115.

[8] G. Bredon, *Topology and Geometry*, Springer-Verlag, New York, 1993.
[9] M. Hirsch, *Differential Topology*, Springer-Verlag, New York, 1976.
[10] D. Hoffman, H. Rosenberg, and J. Spruck, *Boundary value problems for surfaces of constant Gauss curvature*, Comm. Pure. Appl. Math. **45**(1992), 1051–1062.
[11] W. Meeks III, *Lectures on Plateau's problem*, Inst. Mat. Pura Apl., Rio de Jaeiro, 1978.
[12] D. Gilbarg, and N. Trudinger, *Elliptic partial differential equations of second order*, Second edition, Springer-Verlag, Berlin-New York, 1983.
[13] R. Schoen, *Uniqueness, symmetry, and embeddedness of minimal surfaces*, J. Differential Geom. **18** (1983), no. 4, 791–809 (1984).
[14] D. Hoffman, and W. Meeks III, *Minimal Surfaces based on the catenoid*, Amer. Math. Monthly 97 (1990), no. **8**, 702–730.
[15] W. Meeks III, and B. White, *Minimal surfaces bounded by a pair of convex planar curves*, Bull. Amer. Math. Soc. (N.S.) **24** (1991), no. 1, 179–184.
[16] B. Nelli, *An example of an immersed complete genus one minimal surface in R^3 with two convex ends*, Comment. Math. Helv. **73** (1998), no. 2, 298–305.
[17] A. Ros, *Embedded minimal surfaces: forces, topology and symmetries. Calc. Var. Partial Differential Equations*, **4** (1996), no. 5, 469–496.
[18] P. Griffiths, and J. Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York, 1978.
[19] H. Hopf, *Differential geometry in the large*, Lecture Notes in Mathematics, 1000. Springer-Verlag, Berlin-New York, 1989.
[20] S. S. Chern, *Curves and surfaces in Euclidean space*, Studies in Global Geometry and Analysis, vol. 4, Math. Assoc. Amer. (distributed by Prentice-Hall, Englewood Cliffs, N.J.), 1967.
[21] J. van Heijenoort, *On locally convex manifolds*, Comm. Pure Appl. Math., **5**(1952), 223–242.
[22] L. Rodriguez, *Convexity and tightness of manifolds with boundary*, Geometry and topology (Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976), pp. 510–541. Lecture Notes in Math., Vol. 597, Springer, Berlin, 1977.
[23] M. Ghomi, PhD Thesis, Johns Hopkins University, 1998.
[24] ——, *Strictly convex submanifolds and hypersurfaces of positive curvature*, preprint available at `www.math.sc.edu/~ghomi`.
[25] S. Alexander, and M. Ghomi, *The convex hull property and topology of hypersurfaces with nonnegative curvature*, preprint available at `www.math.sc.edu/~ghomi`.

Department of Mathematics, University of South Carolina, Columbia SC 29208
E-mail address: `ghomi@math.sc.edu`
URL: `www.math.sc.edu/~ghomi`