NORMAL CURVATURES OF ASYMPTOTICALLY CONSTANT GRAPHS AND CARATHÉODORY'S CONJECTURE

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ABSTRACT. We show that Carathéodory's conjecture, on umbilical points of closed convex surfaces, may be reformulated in terms of the existence of at least one umbilic in the graphs of functions $f\colon \mathbf{R}^2 \to \mathbf{R}$ whose gradient decays uniformly faster than $1/r$. The divergence theorem then yields a pair of integral equations for the normal curvatures of these graphs, which establish some weaker forms of the conjecture. In particular, we show that there are uncountably many principal lines in the graph of f whose projections into $\mathbf{R}^2$ are parallel to any given direction.

1. INTRODUCTION

Carathéodory's celebrated conjecture [2, 32] which has been the subject of numerous investigations since the 1920s [29], asserts that every (sufficiently smooth) closed convex surface M in Euclidean space $\mathbf{R}^3$ has at least two *umbilics*, i.e., points where the principal curvatures of M are equal. Almost all attempts to prove this claim have been concerned with establishing the more general local conjecture of Loewner on the index of the singularities of principal line fields. Here, by contrast, we develop a global approach to this problem. To state our main results, let us say that a function $f\colon \mathbf{R}^2 \to \mathbf{R}$ is *asymptotically constant* (at a sufficiently fast and uniform rate) provided that, in polar coordinates,

$$\lim_{r\to\infty} f(r,\theta) = const. \quad \text{and} \quad \lim_{r\to\infty} r\,\|\nabla f(r,\theta)\| = 0,$$

uniformly with respect to θ. In other words f converges uniformly to a constant at infinity, while the norm of its gradient decays faster than $1/r$ uniformly in all directions.

Theorem 1.1. *For every $\mathcal{C}^2$ closed convex surface $M \subset \mathbf{R}^3$ and umbilical point $p \in M$, there exists an asymptotically constant $\mathcal{C}^2$ function $f\colon \mathbf{R}^2 \to \mathbf{R}$, and a diffeomorphism between $M \smallsetminus \{p\}$ and $\operatorname{graph}(f)$ which preserves the principal directions. In particular, $\operatorname{graph}(f)$ has one fewer umbilic than M has.*

Thus Carathéodory's conjecture may be reformulated in terms of existence of at least one umbilical point in the graphs of asymptotically constant functions $f\colon \mathbf{R}^2 \to$

R. We use this setting to obtain some evidence in support of the conjecture as stated in the next two results. For any point p in the plane $\mathbf{R}^2$ and direction (or unit vector) X in the circle $\mathbf{S}^1$, let $k(p, X)$ be the *normal curvature* of the graph of f at the point $(p, f(p))$ in the direction of a tangent vector whose projection into $\mathbf{R}^2$ is parallel to X, i.e., set

$$(1) \qquad k(p, X) := \frac{f_{XX}(p)}{\left(1 + f_X^2(p)\right)\sqrt{1 + \|\nabla f(p)\|^2}},$$

where $f_X := \langle \nabla f, X \rangle$ is the derivative of f in the direction X, $f_{XX} := (f_X)_X$, and f_X^2 denotes $(f_X)^2$. Note that f has an umbilic at p (or $(p, f(p))$ is an umbilical point of $\mathrm{graph}(f)$) if $k(p, \cdot)$ is constant on $\mathbf{S}^1$. So, if Carathéodory's conjecture were true, then for every pair of directions X, Y there would exist a point p such that $k(p, X) = k(p, Y)$. The next result shows that there is always an abundant supply of such points for any given pair of directions. Here B_r denotes a closed ball of radius r centered at the origin of $\mathbf{R}^2$, and

$$d\mathcal{A}_p := \sqrt{1 + \|\nabla f(p)\|^2} \, dx \wedge dy$$

is the area element of the graph of f.

Theorem 1.2. *Let $f\colon \mathbf{R}^2 \to \mathbf{R}$ be an asymptotically constant $\mathcal{C}^2$ function. Then for every pair of directions X, $Y \in \mathbf{S}^1$,*

$$\lim_{r\to\infty} \int_{p \in B_r} \left(k(p, X) - k(p, Y)\right)\left(1 + f_X^2(p)\right)\left(1 + f_Y^2(p)\right) d\mathcal{A}_p = 0.$$

In particular the function $k(\,\cdot\,, X) - k(\,\cdot\,, Y)$ is either identically zero or else changes sign on $\mathbf{R}^2$.

To describe our next result, let us say that $X \in \mathbf{S}^1$ is a principal direction of f at $p \in \mathbf{R}^2$ provided that X is parallel to the projection of a principal direction of $\mathrm{graph}(f)$ at $(p, f(p))$ into $\mathbf{R}^2$. Next set $X(\theta) := (\cos(\theta), \sin(\theta))$ and note that $X(\theta_0)$ is a principal direction of f if and only if θ_0 is a critical point of the function $\theta \mapsto k(p, X(\theta))$, for some p. Thus if Carathéodory's conjecture holds, then for every direction $X(\theta_0)$ there should exist a point p such that $\frac{\partial k}{\partial \theta}(p, X(\theta_0)) = 0$. Again, we show that there is no shortage of such points:

Theorem 1.3. *Let $f\colon \mathbf{R}^2 \to \mathbf{R}$ be an asymptotically constant $\mathcal{C}^2$ function. Then for every direction $X(\theta_0) \in \mathbf{S}^1$*

$$\lim_{r\to\infty} \int_{p \in B_r} \frac{\partial k}{\partial \theta}\left(p, X(\theta_0)\right)\left(1 + f_{X(\theta_0)}^2(p)\right) d\mathcal{A}_p = 0.$$

In particular, the set of points $p \in \mathbf{R}^2$ where $X(\theta_0)$ is a principal direction of f at p is either all of $\mathbf{R}^2$ or else separates $\mathbf{R}^2$.

Note that each of the last two results gives considerably more information than can be subsumed by a positive resolution of Carathéodory's conjecture. The last theorem, for instance, guarantees the existence of uncountably many principal lines in the graph of f whose projection into $\mathbf{R}^2$ are parallel to any given direction,

while the existence of an umbilic in the graph of f ensures only the existence of one such line for each direction. Also note that to prove Carathéodory's conjecture it suffices to show that the zero sets of $k(\,\cdot\,, X) - k(\,\cdot\,, Y)$ and $\frac{\partial k}{\partial \theta}(\cdot, X)$, which are abundant by Theorems 1.2 and 1.3, have nonempty intersection for some pair of linearly independent directions $X, Y \in \mathbf{S}^1$; see Note 3.3 for more on this approach. Theorem 1.1 is proved using a combination of Möbius inversions and the operation of moving a surface parallel to itself (Section 2), while Theorems 1.2 and 1.3 are fairly quick applications of the divergence theorem after some computations (Section 3). We will also discuss how to construct closed surfaces, including some convex ones, which are smooth and umbilic free in the complement of one point, and may be arbitrarily close to a sphere (Section 4).

According to Struik [29], the earliest references to the conjecture attributed to Carathéodory appear in the works of Cohn-Vossen, Blaschke, and Hamburger dating back to 1922. The first significant results on the conjecture were due to Hamburger, who established the analytic case in a series of long papers [13, 14, 15] published in 1940–41. Attempts to find shorter proofs attracted the attention of Bol [3], Klotz [18], and Titus [30] in the ensuing decades. As late as 1993, however, Scherbel [23] was still correcting some errors in these simplifications, while reconfirming the validity of Hamburger's theorem. Another reexamination of the proof of the analytic case appears in a comprehensive paper of Ivanov [17] who supplies his own arguments for clarifying various details. All the works mentioned thus far have been primarily concerned with establishing the analytic version of the conjecture of Loewner, which states that the index of the singularities of principal line fields is at most one, and thus implies Carathéodory's conjecture via Poincaré-Hopf index theorem. See Smyth and Xavier [25, 26, 27] for studies of Loewner's conjecture in the smooth case, and Lazarovici [20] for a global result on principal foliations. Another global result is by Feldman [4], who showed that generic closed convex surfaces have four umbilics; also see [9] for some applications of the h-principle to studying homotopy classes of principal lines. A global generalization of Carathéodory's conjecture is discussed in [5], and an interesting analogue of the conjecture for noncompact complete convex hypersurfaces has been studied by Toponogov [31]. A number of approaches to proving the Carathéodory or Loewner conjecture in the smooth case are discussed in [22, 21, 10], and more references or background may be found in [28, 12].

2. Möbius Inversions and Parallel Surfaces: Proof of Theorem 1.1

The basic idea for proving Theorem 1.1 is to blow up an umbilic point of M via a Möbius inversion; however, for the resulting surface to be an asymptotically constant graph, we first need to transform M to a positively curved surface which is close to a sphere with respect to the Whitney $\mathcal{C}^1$-topology. These preliminary transformations are obtained by taking a Möbius inversion of an outer parallel surface of M at a sufficiently large distance, as described below. Let us begin by recording that the

Möbius inversion of any closed set $A \subset \mathbf{R}^3$ is given by

$$m(A) := \mathrm{cl}\left\{ \frac{p}{\|p\|^2} \,\Big|\, p \in A \smallsetminus \{o\} \right\},$$

where cl denotes the closure in $\mathbf{R}^3$ and o is the origin. It is well-known that m *preserves the principal directions* of $\mathcal{C}^2$ surfaces $M \subset \mathbf{R}^3$, i.e., a tangent vector $X \in T_p M$ is a principal direction of M if and only if $dm_p(X)$ is a principal direction of $m(M)$ at $m(p)$, where d denotes the differential map. This follows for instance from the way the second fundamental form is transformed under a conformal change of the metric in the ambient space, e.g., see [16, Lemma P.6.1].

Lemma 2.1. *Let $U \subset \mathbf{R}^2$ be an open neighborhood of o, and $f \colon U \to \mathbf{R}$ be a $\mathcal{C}^1$ function with $f(o) = \|\nabla f(o)\| = 0$. Then there exist $r_0 > 0$, a bounded open neighborhood $V \subset \mathbf{R}^2$ of o, and a function $\overline{f} \colon \mathbf{R}^2 \smallsetminus V \to \mathbf{R}$, such that*

$$m\big(\mathrm{graph}(f|_{B_{r_0}}) \big) = \mathrm{graph}(\overline{f}).$$

Furthermore, if f is $\mathcal{C}^2$, has positive curvature, and an umbilic at o, then $\overline{f}$ is asymptotically constant.

Proof. **(I)** Since f is $\mathcal{C}^1$, and $\|\nabla f(o)\| = 0$, we may choose r_0 so small that, in polar coordinates,

$$(2) \qquad\qquad |f(r_1, \theta) - f(r_2, \theta)| < |r_1 - r_2|,$$

for all $0 \le r_1, r_2 < r_0$ and $\theta \in \mathbf{R}/2\pi$. We claim then that $m(\mathrm{graph}\, f|_{B_{r_0}})$ intersects every vertical line at most once. If this claim holds, then the projection of $m(\mathrm{graph}\, f|_{B_{r_0}})$ into $\mathbf{R}^2$ defines a closed set $\mathbf{R}^2 \smallsetminus V$, where V is some bounded open neighborhood of o, and the height of $m(\mathrm{graph}\, f|_{B_{r_0}})$ over the xy-plane yields the desired function $\overline{f} \colon \mathbf{R}^2 \smallsetminus V \to \mathbf{R}$. To establish our claim, let C be the collection of all circles $c \subset \mathbf{R}^3$ which pass through o and are tangent to the z-axis. Further note that every vertical line coincides with the Möbius inversion of a circle $c \in C$. So it suffices to show that each circle $c \in C$ intersect $\mathrm{graph}(f|_{B_{r_0}})$ at most at one point other than o. To see this suppose, towards a contradiction, that there exists some circle $c \in C$ which intersects $\mathrm{graph}(f|_{B_{r_0}})$ at two distinct points p_1, p_2 other than o. Then $p_i = (r_i, \theta_0, f(r_i, \theta_0))$ in cylindrical coordinates for some fixed $\theta_0 \in \mathbf{R}/2\pi$. Set $f(r) := f(r, \theta)$. Then (2) yields that

$$|f(r_i)| = |f(r_i) - f(o)| < r_i.$$

After a rescaling, we may assume that c has radius one. Then,

$$|f(r_i)| = \sqrt{1 - (1 - r_i)^2} = \sqrt{r_i(2 - r_i)}.$$

The last two displayed expressions yield that $r_i > 1$. On the other hand, by (2) and the triangle inequality,

$$|r_2 - r_1| \ge |f(r_2) - f(r_1)| \ge |f(r_2)| - |f(r_1)| \ge \sqrt{r_2(2 - r_2)} - r_1.$$

So if we suppose that $r_2 \ge r_1$, the last expression shows that $r_2 < 1$ and we have a contradiction.

(II) Now suppose that f is $\mathcal{C}^2$, has an umbilic at o, and is positively curved at o. To show that f is asymptotically constant we proceed in three stages:

(II.1) First we derive some estimates for f and its partial derivatives f_r and f_θ in polar coordinates. Since principal curvatures of f are equal and positive at o, we may assume after a rescaling that $f - r^2$ vanishes up to order 2 at o. So it follows that $\lim_{r\to 0}(f - r^2)/r^2 = 0$. Consequently the function ρ given by $\rho(o) := 0$, and $\rho(x,y) := (f - r^2)/r^2$ for $(x,y) \neq o$ is continuous, and yields our first estimate:

$$(3) \qquad f = r^2\left(1 + \frac{f - r^2}{r^2}\right) = r^2(1 + \rho).$$

Similarly, since $f - r^2$ vanishes up to second order at o, it follows that $f_x - 2x$ and $f_y - 2y$ vanish up to the first order at o, which in turn yields that $f_x = 2x + r\xi$ and $f_y = 2y + r\eta$, for some continuous functions ξ and η which vanish at o. Using the chain rule, we now compute that

$$(4) \quad f_r = \frac{xf_x + yf_y}{r} = \frac{x(2x + r\xi) + y(2x + r\eta)}{r} = 2r + \left(\frac{x}{r}\xi + \frac{y}{r}\eta\right)r = r(2 + \phi),$$

where $\phi(o) := 0$ and $\phi(x,y) := (x/r)\xi + (y/r)\eta$ for $(x,y) \neq o$. Since $|x/r|, |y/r| \leq 1$, it follows that $|\phi| \leq |\xi| + |\eta|$, which shows that ϕ is continuous and vanishes at o. Likewise

$$(5) \qquad f_\theta = -yf_x + xf_y = -y(2x + r\xi) + x(2y + r\eta) = \left(-\frac{y}{r}\xi + \frac{x}{r}\eta\right)r^2 = r^2\psi,$$

where ψ is again continuous and vanishes at o.

(II.2) Next we show that $\overline{f}$ converges uniformly to a constant at infinity. With r_0 as in part (I) and for every $p \in B_{r_0}$, set

$$\left(\overline{p}, \overline{f}(\overline{p})\right) := m\left((p, f(p))\right) = \left(\frac{p}{\|p\|^2 + f^2(p)}, \frac{f(p)}{\|p\|^2 + f^2(p)}\right).$$

Thus if $p = (r,\theta)$ and $\overline{p} = (\overline{r}, \overline{\theta})$ in polar coordinates, then $\overline{\theta} = \theta$, and (3) yields that

$$(6) \qquad \overline{r} = \frac{r}{r^2 + f^2(r,\theta)} = \frac{1}{r + r^3(1 + \rho)^2},$$

$$(7) \qquad \overline{f}(\overline{r}, \theta) = \frac{f(r,\theta)}{r^2 + f^2(r,\theta)} = \frac{1 + \rho}{1 + r^2(1 + \rho)^2}.$$

Choosing r_0 sufficiently small, we may assume that $r^3(1 + \rho)^2 < r$ for $r < r_0$. Then (6) yields $1/2r < \overline{r} < 1/r$. So $\overline{r} \to \infty$ if and only if $r \to 0$. Now by (7)

$$\lim_{\overline{r}\to\infty} \overline{f}(\overline{r}, \theta) = \lim_{r\to 0} \frac{1 + \rho}{1 + r^2(1 + \rho)^2} = 1,$$

and the continuity of ρ ensures that the convergence is uniform (with respect to θ).

(II.3) It remains only to check the rate of decay of $\nabla\overline{f}$. First note that by (6)

$$(8) \qquad \lim_{r\to 0} r\overline{r} = \lim_{\overline{r}\to\infty} r\overline{r} = \lim_{\overline{r}\to\infty} \frac{1}{1 + r^2(1 + \rho)^2} = 1,$$

where the continuity of ρ again ensures the uniformity of the convergence. We need to show that $\lim_{\bar{r}\to\infty} \bar{r}\|\nabla\overline{f}(\bar{r},\theta)\| = 0$ uniformly. Thus, since $\nabla\overline{f} = (\overline{f}_{\bar{r}}, \overline{f}_\theta/\bar{r})$ in polar coordinates, it suffices to check that

$$\lim_{\bar{r}\to\infty} \bar{r}\overline{f}_{\bar{r}}(\bar{r},\theta) = 0, \quad \text{and} \quad \lim_{\bar{r}\to\infty} \overline{f}_\theta(\bar{r},\theta) = 0,$$

uniformly. To establish the first convergence note that, by the chain rule and estimates (3) and (4)

$$\bar{r}\overline{f}_{\bar{r}} = \bar{r}\,\frac{\partial\overline{f}/\partial r}{\partial\bar{r}/\partial r} = \bar{r}\,\frac{r^2 f_r - 2rf - f^2 f_r}{-r^2 + f^2 - 2rff_r} = r\bar{r}\,\frac{\phi - 2\rho - r^2(1+\rho)^2(2+\phi)}{-1 + r^2(1+\rho)^2 - 2r^2(1+\rho)(2+\phi)}.$$

Thus by (8), and since $\rho(o) = \phi(o) = 0$,

$$\lim_{\bar{r}\to\infty} \bar{r}\overline{f}_{\bar{r}} = \lim_{r\to 0} \frac{\phi - 2\rho - r^2(1+\rho)^2(2+\phi)}{-1 + r^2(1+\rho)^2 - 2r^2(1+\rho)(2+\phi)} = 0,$$

where the convergence is uniform by the continuity of ρ and ϕ. Similarly, (7) together with estimates (3) and (5) yield that

$$\lim_{\bar{r}\to\infty} \overline{f}_\theta = \lim_{r\to 0} \frac{(r^2 - f^2)f_\theta}{(r^2 + f^2)^2} = \lim_{r\to 0} \frac{(1 - r^2(1+\rho)^2)\psi}{(1 + r^2(1+\rho)^2)^2} = 0,$$

since $\psi(o) = 0$, and the convergence is once again uniform by the continuity of ρ and ψ. $\qquad\square$

The next lemma we need employs the notion of an outer parallel surface, which is defined as follows. A *closed convex surface* M is the boundary of a compact convex set with interior points in $\mathbf{R}^3$. If M is $\mathcal{C}^{k\geq 1}$, and $n\colon M \to \mathbf{S}^2$ is its outward unit normal vector field or Gauss map, then for any $r \geq 0$, the *outer parallel surface* of M at the distance r is defined as

$$M^r := \{\, p + rn(p) \mid p \in M \,\},$$

which is again a $\mathcal{C}^k$ surface [6, 19]. Furthermore it is easy to show that the mapping $f\colon M \to M^r$ given by $f(p) := p + rn(p)$ preserves the principal directions (assuming M is $\mathcal{C}^2$). To see this, let $X \in T_pM$ be a principal direction of M, then $dn_p(X) = k(p)X$. So $df_p(X) = (1 + rk(p))X$. In particular, T_pM and $T_{f(p)}M^r$ are parallel. Thus $n^r(f(p)) = n(p)$ where $n^r\colon M^r \to \mathbf{S}^2$ is the Gauss map of M^r. Now we may compute that

$$dn^r_{f(p)}\Big(df_p(X)\Big) = d(n^r \circ f)_p(X) = dn_p(X) = k(p)X = \frac{k(p)}{1 + rk(p)}df_p(X),$$

which shows f preserves principal directions as claimed. Next, we also need to recall that the space of $\mathcal{C}^1$ maps $\mathbf{S}^2 \to \mathbf{R}^3$ carries the *Whitney $\mathcal{C}^1$-topology*, which may be defined by stipulating that a pair of $\mathcal{C}^1$ mappings $f, g\colon \mathbf{S}^2 \to \mathbf{R}^3$ are $\mathcal{C}^1$-*close* provided that $\|f - g\| \leq \varepsilon$ and $\|f_i - g_i\| \leq \varepsilon$, $i = 1, 2$, with respect to some atlas of local coordinates on $\mathbf{S}^2$. A pair of embedded spheres are said to be $\mathcal{C}^1$-close if their inclusion maps are $\mathcal{C}^1$-close.

Lemma 2.2. *For every $\mathcal{C}^2$ closed convex surface $M \subset \mathbf{R}^3$, there exists a $\mathcal{C}^2$ closed positively curved surface M' which may be arbitrarily $\mathcal{C}^1$-close to $\mathbf{S}^2$, and a diffeomorphism $M \to M'$ which preserves the principal directions.*

Proof. Since M is $\mathcal{C}^2$ it has an inner support ball at each point by Blaschke's rolling theorem [24]. Indeed, if we let ρ be the minimum of the radii of curvature of M, then through each point p of M there passes a ball B_p of radius ρ which lies inside the convex body bounded by M. Suppose that there exists a common point o in the interior of all these inner support balls. Then through each point $m(p)$ of the Möbius inversion $m(M)$ there passes a ball $m(B_p)$ which contains $m(M)$. Hence $m(M)$ has strictly positive curvature, and is the desired surface M', since m preserves the principal directions. If, on the other hand, the inner support balls of M do not have a common point, we may replace M by an outward parallel surface M^r at a sufficiently large distance r. Then the inner support balls of M^r will pass through a common point, and it remains only to recall that the mapping $M \to M^r$ given by $p \mapsto p + rn(p)$ preserves the principal directions, as we demonstrated above. Finally, choosing r sufficiently large, we may make sure that M^r is as $\mathcal{C}^1$-close to a sphere as desired. Indeed $M_r/(1+r)$ converges to a sphere S with respect to the Whitney $\mathcal{C}^1$-topology as $r \to \infty$, and after a rigid motion and rescaling we may assume that $S = \mathbf{S}^2$. $\qquad\square$

We only need one more basic fact for the proof of Theorem 1.1. A *round sphere* is any embedding $\mathbf{S}^2 \to \mathbf{R}^3$ which is obtained by a rigid motion and homothety of the standard sphere.

Lemma 2.3. *Let $M \subset \mathbf{R}^3$ be a $\mathcal{C}^1$ embedded sphere which is tangent to the xy-plane at o. If M is sufficiently $\mathcal{C}^1$-close to a round sphere, then the Möbius inversion of M forms a graph over the xy-plane.*

Proof. The overall strategy here is similar to that in the proof of Lemma 2.3, i.e., we just need to check that every circle $c \in C$ intersects M at precisely two points, where C again stands for the space of all circles in $\mathbf{R}^3$ which are tangent to the z-axis at o. To this end first recall that, as we showed in the proof of Lemma 2.1, there exists an $\varepsilon > 0$ such that circles $c \in C$ of radius $\leq \varepsilon$ intersect M at precisely two points already. So it suffices to consider only the collection of circles $C_{\geq \varepsilon/2} \subset C$ of radius $\geq \varepsilon/2$. Next note that by assumption M is $\mathcal{C}^1$-close to a sphere S, which we may assume to be tangent to the xy-plane at o. Further every circle $c \in C$ intersects S orthogonally. Since M and S are $\mathcal{C}^1$-close, we may conclude then that M intersects every circle $c \in C_{\geq \varepsilon/2}$ transversely. Thus it follows that the number $\#(c \cap M)$ of intersection points of c with M is finite, and the mapping $c \mapsto \#(c \cap M)$ is locally constant on $C_{\geq \varepsilon/2}$ with respect to the Hausdorff topology on C. This yields that $c \mapsto \#(c \cap M)$ is a constant function on $C_{\geq \varepsilon/2}$, since $C_{\geq \varepsilon/2}$ is connected. So $\#(c \cap M) = 2$ for every $c \in C_{\geq \varepsilon/2}$, since circles of radius ε intersects M precisely twice. $\qquad\square$

Now we have all the pieces in place to complete the proof of Theorem 1.1, as follows. After a rigid motion we may assume that an umbilical point p of M lies

at the origin o of $\mathbf{R}^3$, and M is tangent to the xy-plane at o. Further, by Lemma 2.2 we may assume that M is $\mathcal{C}^1$-close to a sphere and is positively curved. So the Möbius inversion $m(M)$ forms a graph over the xy-plane by Lemma 2.3. Finally, this graph is asymptotically constant by Lemma 2.1.

Note 2.4. The proof of Theorem 1.1 in this section reveals that the theorem is valid not just for convex surfaces, but applies to any $\mathcal{C}^2$ closed surface $M \subset \mathbf{R}^3$ which may be transformed to a convex one by means of Möbius transformations, and the operation of moving M parallel to itself.

Note 2.5. In Lemma 2.2 we may choose the surface M' to be $\mathcal{C}^2$-close to M. Indeed, as we showed in the proof of the lemma, we may assume that M has positive curvature. Then the *support function* $h_M \colon \mathbf{R}^2 \to \mathbf{R}$ of M is also $\mathcal{C}^2$, e.g., see [24]. Further, the support function of the outer parallel surface M^r is $r + h_M$, so if we dilate M^r by the factor $1/r$, then

$$h_{\frac{1}{r}M^r} = 1 + \frac{h_M}{r}$$

which converges to 1 in the $\mathcal{C}^2$ topology. Thus $(1/r)M^r$ converges to $\mathbf{S}^2$ in the $\mathcal{C}^2$ topology. This argument also shows that if M is $\mathcal{C}^k$, $2 \leq k \leq \infty$, then $(1/r)M^r$ converges to $\mathbf{S}^2$ in the $\mathcal{C}^k$ topology.

3. Applications of the Divergence Theorem: Proofs of Theorems 1.2 and 1.3

Let us say that a function $f \colon \mathbf{R}^2 \to \mathbf{R}$ *decays uniformly faster* than $1/r$ provided that, in polar coordinates, $\lim_{r\to\infty} rf(r,\theta) = 0$ uniformly in θ. Here $\nabla\cdot$ stands for divergence.

Lemma 3.1. *Let $V \colon \mathbf{R}^2 \to \mathbf{R}^2$ be a $\mathcal{C}^1$ vector field whose norm decays uniformly faster than $1/r$. Then*

$$\lim_{r\to\infty} \int_{B_r} \nabla \cdot V = 0.$$

Proof. Let $c(\theta) := (r\cos(\theta), r\sin(\theta))$ be a parametrization for ∂B_r, and n be its outward unit normal. By the divergence theorem and the Cauchy-Schwartz inequality

$$\left| \int_{B_r} \nabla \cdot V \right| = \left| \int_{\partial B_r} V \cdot n \right|$$

$$\leq \int_{\partial B_r} \|V\| = \int_0^{2\pi} \|V(c(\theta))\| \|c'(\theta)\| d\theta = \int_0^{2\pi} \|V(r,\theta)\| r \, d\theta.$$

Thus it follows that

$$\lim_{r\to\infty} \left| \int_{B_r} \nabla \cdot V \right| \leq \lim_{r\to\infty} \int_0^{2\pi} \|V(r,\theta)\| r d\theta = \int_0^{2\pi} \lim_{r\to\infty} \|V(r,\theta)\| r d\theta = 0,$$

where exchanging the order of integration and the limit here is warranted by the assumption that the convergence of the functions $\|V(r,\theta)\| r$ is uniform in θ. $\qquad \square$

Equipped with the last fact, we are now ready to complete the proofs of our main results:

Proof of Theorem 1.2. Set $X = (X^1, X^2)$, $Y = (Y^1, Y^2)$, and note that $f_{XY} = \sum_{i,j=1}^2 f_{ij} X^i Y^j = f_{YX}$. Thus, if $k_X(p) := k(p, X)$, then using (1) we have

$$(k_X - k_Y)(1 + f_X^2)(1 + f_Y^2)\sqrt{1 + \|\nabla f(p)\|^2}$$
$$= f_{XX}(1 + f_Y^2) - f_{YY}(1 + f_X^2)$$
$$= \left(f_X(1 + f_Y^2)\right)_X - \left(f_Y(1 + f_X^2)\right)_Y.$$

Setting $u := f_X(1 + f_Y^2)$, $v := f_Y(1 + f_X^2)$, we continue the above computation as:

$$u_X - v_Y = u_1 X^1 + u_2 X^2 - v_1 Y^1 - v_2 Y^2$$
$$= \left(uX^1 - vY^1\right)_1 + \left(uX^2 - vY^2\right)_2.$$

The last line is the divergence of the vector field

$$(uX^1 - vY^1, uX^2 - vY^2),$$

whose norm vanishes uniformly faster than $1/r$ since f is asymptotically constant. So applying Lemma 3.1 to this vector field completes the proof. $\square$

Proof of Theorem 1.3. After a rotation of the coordinate axis we may assume that $\theta_0 = 0$. A simple computation shows that

$$k_{X(\theta)} = \frac{f_{11}\cos^2(\theta) + f_{12}\sin(2\theta) + f_{22}\sin^2(\theta)}{1 + \left(f_1\cos(\theta) + f_2\sin(\theta)\right)^2} \frac{1}{\sqrt{1 + \|\nabla f\|^2}},$$

and another straight forward calculation yields

$$\frac{\partial}{\partial\theta}\Big|_{\theta=0} k_{X(\theta)} = \frac{2\left((1 + f_1^2)f_{12} - f_2 f_1 f_{11}\right)}{\left(1 + f_1^2\right)^2} \frac{1}{\sqrt{1 + \|\nabla f\|^2}}.$$

So $(1, 0) = X(0)$ is a principal direction of f at p if and only if the following equation holds at p

$$(1 + f_1^2)f_{12} - f_2 f_1 f_{11} = 0,$$

which is equivalent to

$$(9) \qquad \left(\frac{f_2}{\sqrt{1 + f_1^2}}\right)_1 = 0.$$

Thus applying Lemma 3.1 to the vector field $(f_2/\sqrt{1 + f_1^2}, 0)$ completes the proof.
$\square$

Note 3.2. The proofs of Theorems 1.2 and 1.3 do not use the assumption that f converges to a constant at infinity. Indeed these results require only that the gradient vanishes uniformly faster than $1/r$. So f need not even be bounded (consider for instance any smooth function $f: \mathbf{R}^2 \to \mathbf{R}$ which coincides with $\ln(\ln r)$ outside a compact set).

Note 3.3. A surface $M \subset \mathbf{R}^3$ has an umbilical point p if and only if its second fundamental form is a multiple of its first fundamental form at p. When M is the graph of a function $f \colon \mathbf{R}^2 \to \mathbf{R}$, this means that

$$\operatorname{rank} \begin{pmatrix} f_{11} & f_{12} & f_{22} \\ 1 + f_1^2 & f_1 f_2 & 1 + f_2^2 \end{pmatrix} \le 1,$$

which is equivalent to the following system of equations:

$$(10) \qquad\qquad (1 + f_1^2) f_{12} - f_1 f_2 f_{11} = 0,$$
$$(1 + f_1^2) f_{22} - (1 + f_2^2) f_{11} = 0.$$

Indeed the first equation holds if and only if $(1,0)$ is a principal direction of f, as we showed in the proof of Theorem 1.3, and the second equation holds if and only if the normal curvatures of f in the directions $(1,0)$ and $(0,1)$ agree, as we saw in the proof of Theorem 1.2. Also recall that the above system of equations is equivalent to

$$\left(\frac{f_2}{\sqrt{1 + f_1^2}} \right)_1 = 0,$$
$$\left((1 + f_1^2) f_2 \right)_2 - \left((1 + f_2^2) f_1 \right)_1 = 0.$$

Yet another way to characterize the umbilics of a graph is as the solutions of

$$\left(f_{22} \left(1 + f_1^2 \right) - 2 f_1 f_2 f_{12} + f_{11} \left(1 + f_2^2 \right) \right)^2 - 4 \left(1 + f_1^2 + f_2^2 \right) \left(f_{22} f_{11} - f_{12}^2 \right) = 0.$$

The above PDE is obtained by setting $H^2 - K$ equal to zero, where H and K are the mean and Gauss curvatures of the graph of f respectively. This also yields a coordinate free expression which is equivalent to the equations above:

$$\left(\nabla \cdot \left(\frac{\nabla f}{\sqrt{1 + \|\nabla f\|^2}} \right) \right)^2 - \frac{4 \det \operatorname{Hess}(f)}{(1 + \|\nabla f\|^2)^2} = 0.$$

To prove Carathéodory's conjecture it suffices to show that any one of the above systems holds at some point, assuming f is asymptotically constant.

4. Some Examples

4.1.   The graph of a function $f \colon \mathbf{R}^2 \to \mathbf{R}$ which is not asymptotically constant may not have any umbilics. Indeed, any negatively curved graph such as $f(x,y) = xy$, or any cylindrical graph given by $f(x,y) = g(x)$ where $g \colon \mathbf{R} \to \mathbf{R}$ is a smooth function without inflection points are umbilic free. There are even umbilic free graphs which are bounded above and below. One such example is due to Bates [1], and another is given by

$$f_\lambda(x,y) = 1 + \lambda \frac{x + y^2}{\sqrt{1 + (x + y^2)^2}},$$

for $\lambda > 0$. Indeed, with the aid of a computer algebra system, one may easily verify that the equations (10) are never simultaneously satisfied for f_λ, unless $\lambda = 0$. The

Möbius inversion of graph(f_λ) is $\mathcal{C}^\infty$ in the complement of one point, is differentiable everywhere, and converges to a sphere with respect to Hausdorff distance as $\lambda \to 0$; see Figure 1. The differentiability of the inversion is due to the fact that graph(f_λ) is contained between a pair of horizontal planes, which in turn implies that the inversion rests between a pair of spheres at o. In particular, the tangent cone of the inversion at o is a plane.
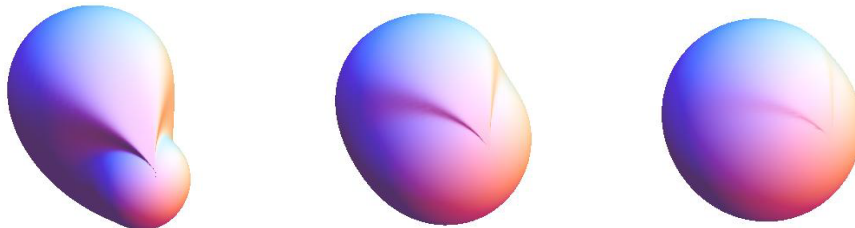

FIGURE 1.

4.2. Using Möbius inversions, one may construct closed *convex* surfaces which are umbilic free and $\mathcal{C}^\infty$ in the complement of one point. A family of such examples is given by inverting the graph of

$$f_\lambda(x, y) = 1 + \lambda \sqrt{1 + x^2},$$

for sufficiently small $\lambda > 0$. It is easy to see that these surfaces are umbilic free for all $\lambda \neq 0$. Further, if λ is small, then each support plane of these graphs separates the graph from the origin o. Thus the inversion of the graph has a supporting sphere at each point which shows that it must be convex. Furthermore, since f_λ converges to the plane $z = 1$, it follows that inversions of graph(f_λ) converge to a sphere as $\lambda \to 0$, as shown in Figure 2. An approximation of these inversions, which may be
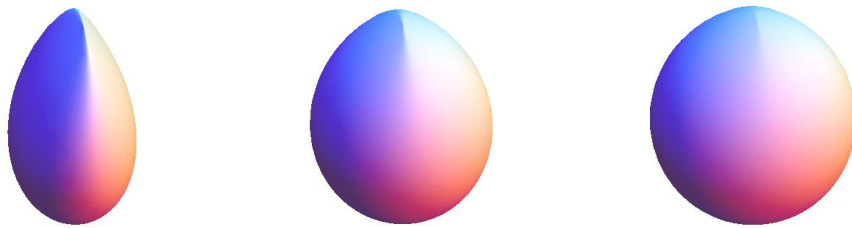

FIGURE 2.

achieved for instance by applying a certain convolution to their support functions [24, 8], yields $\mathcal{C}^\infty$ closed convex surfaces all of whose umbilics are contained in a region with arbitrarily small diameter and total curvature, c.f. [11]. Indeed, a neighborhood of the singular point may be represented as the graph of a convex function over a support plane. Identifying the support plane with the $\mathbf{R}^2 \times \{0\} \subset \mathbf{R}^3$,

we then obtain a convex function $f\colon B_r \to \mathbf{R}$ which is $\mathcal{C}^\infty$ and positively curved in the complement of the origin o. For every open neighborhood U of o, with closure $\overline{U} \subset B_r$, there exists a $\mathcal{C}^\infty$ convex function $\widetilde{f}\colon B_r \to \mathbf{R}$ such that $\widetilde{f} = f$ on $B_r - U$, see [7]. Replacing $\mathrm{graph}(f)$ with $\mathrm{graph}(\widetilde{f})$ then yields a smoothing of our surfaces, which preserves each surface in the complement of any given open neighborhood of the singularity.

4.3.   Another example of a family of convex surfaces which are umbilic free and $\mathcal{C}^\infty$ in the complement of one point is obtained by inverting the graphs of

$$f_\lambda(x,y) = 1 + \lambda \left(\sqrt{1+x^2} + x + \sqrt{1+y^2} + y \right)$$

for $\lambda > 0$. It is easy to show that these graphs never satisfy the first equation in (10) unless $\lambda = 0$. It is also worth remarking that, in contrast to the previous example, these graphs have everywhere positive curvature. Thus one may say that Carathéodory's conjecture has no analogue for complete convex surfaces which are not compact. Similar to the previous example, the inversion of these graphs converge to a sphere as $\lambda \to 0$, see Figure 3; however, here, the singularity is of the "cone

FIGURE 3.

type", i.e., the normal cone at the singularity has dimension 3 (as opposed to the previous example where the singularity was of the "ridge type", i.e., the normal cone was only 2-dimensional). Similar examples may be generated by any function of the form $f_\lambda(x,y) = 1 + \lambda(g(x) + h(y))$ where g and h are $\mathcal{C}^2$, neither g' or h' ever vanishes, and g'' and h'' are always positive.

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