

DIRECTED IMMERSIONS OF CLOSED MANIFOLDS

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ABSTRACT. Given any finite subset X of the sphere \mathbf{S}^n , $n \geq 2$, which includes no pairs of antipodal points, we explicitly construct smoothly immersed closed orientable hypersurfaces in Euclidean space \mathbf{R}^{n+1} whose Gauss map misses X . In particular, this answers a question of M. Gromov.

1. INTRODUCTION

To every (\mathcal{C}^1) immersion $f: M^n \rightarrow \mathbf{R}^{n+1}$ of a closed oriented n -manifold M , there corresponds a unit normal vector field or *Gauss map* $G_f: M \rightarrow \mathbf{S}^n$, which generates a set $G_f(M) \subset \mathbf{S}^n$ known as the *spherical image* of f . Conversely, one may ask, c.f. [8, p. 3]: *for which sets $A \subset \mathbf{S}^n$ is there an immersion $f: M \rightarrow \mathbf{R}^{n+1}$ such that $G_f(M) \subset A$?* Such a mapping would be called an *A -directed immersion* of M [1, 7, 13, 14]. It is well-known that when $A \neq \mathbf{S}^n$, f must have double points (Note 4.1), and M must be parallelizable, e.g., M can only be the torus \mathbf{T}^2 when $n = 2$ (Note 4.2). Furthermore, the only known necessary condition on A is the elementary observation that $A \cup -A = \mathbf{S}^2$, while there is also a sufficient condition due to Gromov [7, Thm. (D'), p. 186]:

Condition 1.1. *$A \subset \mathbf{S}^n$ is open, and there is a point $p \in \mathbf{S}^n$ such that the intersection of A with each great circle passing through p includes a (closed) semicircle.*

A *great circle* is the intersection of \mathbf{S}^n with a 2-dimensional subspace of \mathbf{R}^{n+1} . Note that, when $n \geq 2$, examples of sets $A \subset \mathbf{S}^n$ satisfying the above condition include those which are the complement of a finite set of points without antipodal pairs. Thus the spherical image of a closed hypersurface can be remarkably flexible. Like most h -principle or convex integration type arguments, however, the proof does not yield specific examples. It is therefore natural to ask, for instance:

Question 1.2 ([7], p. 186). *“Is there a ‘simple’ immersion $\mathbf{T}^2 \rightarrow \mathbf{R}^3$ whose spherical image misses the four vertices of a regular tetrahedron in \mathbf{S}^2 ?”*

Here we give an affirmative answer to this question (Section 2), and more generally present a short constructive proof of the sufficiency of a slightly stronger version of Condition 1.1 for the existence of A -directed immersions of parallelizable manifolds

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$M^{n-1} \times \mathbf{S}^1$, where M^{n-1} is closed and orientable. Any such manifold admits an immersion $f: M^{n-1} \rightarrow \mathbf{R}^n \times \{0\} \subset \mathbf{R}^{n+1}$ (Note 4.3). We then extend f to $M^{n-1} \times \mathbf{S}^1$ by using the *figure eight curve*

$$(1) \quad E_\delta(t) := (\cos(t), \delta \sin(2t))$$

to put a copy of $\mathbf{S}^1 \simeq \mathbf{R}/2\pi$ in each normal plane of f , as described below. Note that the midpoint of $G_{E_\delta}(\mathbf{S}^1)$ is assumed to be at $(1, 0)$; see Figure 1 which shows $E_{1/2}$ and its spherical image. Further, the unit normal bundle of f may be naturally identified with the pencil of great circles of \mathbf{S}^n passing through $(0, \dots, 0, 1)$.

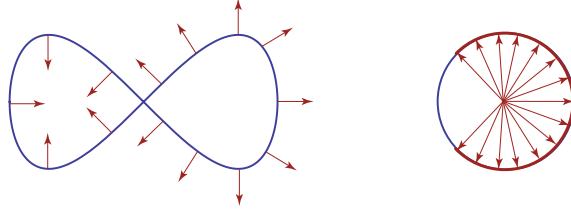


FIGURE 1.

Theorem 1.3. *Let $A \subset \mathbf{S}^n$ satisfy Condition 1.1 with respect to $p = (0, \dots, 0, 1)$. Further, if $n \geq 3$, suppose that the semicircle in Condition 1.1 contains p , or that no great circle through p is contained in A . Let $f: M^{n-1} \rightarrow \mathbf{R}^n \times \{0\} \subset \mathbf{R}^{n+1}$ be a smooth (C^∞) immersion of a closed orientable $(n-1)$ -manifold, and, for every $q \in M$, $C_q \subset \mathbf{S}^n$ be the unit normal space of f at q . Then there is a smooth orthogonal frame $\{N_i: M \rightarrow \mathbf{S}^n\}$, $i = 1, 2$, for the normal bundle of f such that the semicircle in C_q centered at $N_1(q)$ lies in A . For any such frame, and sufficiently small ε , $\delta > 0$,*

$$(2) \quad F(q, t) := f(q) + \varepsilon \sum_{i=1}^2 E_\delta^i(t) N_i(q)$$

yields a smooth A -directed immersion $M \times \mathbf{S}^1 \rightarrow \mathbf{R}^{n+1}$, where E_δ^i are the components of the figure eight curve E_δ given by (1).

It is not known if Condition 1.1 is necessary for the existence of A -directed closed hypersurfaces, and the question posed in the first paragraph is open, even for $n = 2$. See [3, 4, 6] for some other recent results on Gauss maps of closed submanifolds, [2, 9, 11, 16] for still more studies of spherical images, and [15] for historical background.

2. EXAMPLE

If $A = \mathbf{S}^2 \setminus X$ for a finite set X without antipodal pairs, we may always find a point $p \in \mathbf{S}^2$ with respect to which A satisfies the hypothesis of Theorem 1.3 (e.g., let $p \notin X$ be in the complement of all great circles which pass through at least two points of X other than $-p$). After a rigid motion (which may be arbitrarily small)

we may assume that $p = (0, 0, 1)$ or $(0, 0, -1)$, and let $f(\theta) := (\cos(\theta), \sin(\theta), 0)$ be the standard immersion of $\mathbf{S}^1 \simeq \mathbf{R}/2\pi$ in \mathbf{R}^3 . Then the desired framing for the normal bundle of f may always take the form

$$(3) \quad N_1(\theta) := f'(\theta) \times N_2(\theta), \quad N_2(\theta) := \frac{(\cos(\theta), \sin(\theta), z(\theta))}{\sqrt{1 + z^2(\theta)}},$$

where $z: \mathbf{R}/2\pi \rightarrow \mathbf{R}$ is a smooth function with $z(\theta) = -z(\theta + \pi)$ and such that X is contained entirely in one of the components of $\mathbf{S}^2 - N_2(\mathbf{S}^1)$. For instance, when X is the vertices of a regular tetrahedron, we may set $z(\theta) := \cos(3\theta)$ in (3). Then, for $\varepsilon, \delta \leq 1/8$, the mapping $F(\theta, t)$ given by (2) yields an immersion $\mathbf{T}^2 \simeq \mathbf{R}/2\pi \times \mathbf{R}/2\pi \rightarrow \mathbf{R}^3$ which answers Question 1.2. The resulting surface, for $\varepsilon = \delta = 1/8$, is depicted in Figure 2 together with its spherical image (note that

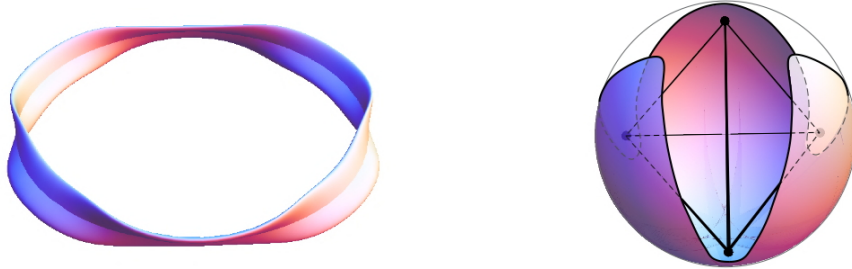


FIGURE 2.

here $p = (0, 0, -1)$). To find $z(\theta)$ in general, we may order the points in $X' \cup -X'$, where $X' := X \setminus \{-p\}$, according to their “longitude” θ , and connect them by geodesic segments to obtain a simple closed symmetric curve $\gamma(\theta)$. A perturbation of γ then yields a smooth symmetric curve $\tilde{\gamma}$ such that X is contained in one of the components of $\mathbf{S}^2 - \tilde{\gamma}(\mathbf{S}^1)$. The third coordinate of $\tilde{\gamma}$ gives our desired height function z .

3. PROOF OF THEOREM 1.3

3.1. First we construct the frame $\{N_i\}$. For every $q \in M$, C_q is a great circle passing through p . So it contains a semicircle in A by assumption (Condition 1.1). Let $m_q \subset C_q$ be the set of midpoints of all such semicircles. We need to find a smooth map $N_1: M \rightarrow \mathbf{S}^n$ such that $N_1(q) \in m_q$ for all $q \in M$. To this end note that m_q is open and connected. Further, if m_q contains any pairs of antipodal points, then $m_q = C_q$; otherwise, m_q lies in the interior a semicircle of C_q . Consequently,

$$\text{Cone}(m_q) := \{ \lambda x \mid x \in m_q, \text{ and } \lambda \geq 0 \},$$

is a convex set in \mathbf{R}^{n+1} . In particular, for any finite set of points $x_i \in \text{Cone}(m_q)$ and numbers $\lambda_i \geq 0$, $\sum_i \lambda_i x_i \in \text{Cone}(m_q)$. Now let B be the set of all points $q \in M$ such that $m_q \neq C_q$. Then B is closed (and therefore compact) since $M \setminus B$ is open; indeed the set of great circles contained in A is open, since A is open. Further note

that for any point $q \in M$, normal vector $x \in m_q$, and continuous local extension v of x to a normal vector field of M , we have $v(q') \in m_{q'}$ for all q' in an open neighborhood U of q (because the set of semicircles contained in A is open). Let $\{v_i: U_i \rightarrow \mathbf{S}^n\}$, $i = 1, \dots, k$, be a finite collection of such local vector fields so that $\cup_i U_i$ covers B and v_i are smooth. Also let $\{\phi_i: M \rightarrow \mathbf{R}\}$ be a smooth partition of unity subordinate to $\{U_i\}$, and, for $q \in \cup_i U_i$, set

$$N_1(q) := \frac{\sum_{i=1}^k \phi_i(q)v_i(q)}{\|\sum_{i=1}^k \phi_i(q)v_i(q)\|}.$$

If $q \in B$, then $v_i(q) \in m_q$ which lies in the interior of a semicircle $S \subset C_q$, and so $\|\sum_{i=1}^k \phi_i(q)v_i(q)\| \neq 0$. Indeed, if x is the midpoint of S , then $\langle \sum_{i=1}^k \phi_i(q)v_i(q), x \rangle = \sum_{i=1}^k \phi_i(q)\langle v_i(q), x \rangle > 0$. Thus N_1 is well defined (and smooth) on an open neighborhood V of B . Further, $N_1(q) \in m_q$, for all $q \in V$, since $\text{Cone}(m_q)$ is convex. In particular we are done if $B = M$; otherwise, note that we may write

$$(4) \quad N_1(q) = \cos(\theta(q))p + \sin(\theta(q))G_f(q),$$

for some function $\theta: V \rightarrow \mathbf{R}$, since G_f is well defined due to the orientability of M , and thus $\{p, G_f(q)\}$ forms an orthonormal basis for the normal plane $df(T_q M)^\perp$. Further, it is easy to see that we may choose θ continuously (and therefore smoothly) if $n = 2$. This also holds for $n > 2$ if each C_q contains a semicircle passing through p ; for then θ is uniquely determined within the range $[-\pi/2, \pi/2]$. Indeed, we may choose the vectors v_i above so that $\langle v_i(q), p \rangle \geq 0$ which would in turn yield that $\langle N_1(q), p \rangle \geq 0$. Now let V' be an open neighborhood of B with closure $\overline{V'} \subset V$. Using Tietze's theorem, followed by a perturbation and a gluing, we may extend $\theta|_{V'}$ smoothly to all of M . Then (4) yields the desired vector field on M , since for any $q \in M \setminus B$, $N_1(q) \in C_q = m_q$. Finally, set

$$N_2(q) := \sin(\theta(q))p - \cos(\theta(q))G_f(q).$$

3.2. It remains to show that $G_F(M \times \mathbf{S}^1) \subset A$, for small ε , $\delta > 0$. For all $q \in M$, $C_q \cap A$ contains an arc of length $\geq \pi + \alpha$ with midpoint $N_1(q)$ for some uniform constant $\alpha > 0$. Indeed, if we let $g(q)$ be the supremum of lengths of all arcs in $C_q \cap A$ with midpoint $N_1(q)$, then $g: M \rightarrow \mathbf{R}$ is lower semicontinuous, i.e., $\lim_{q \rightarrow q_0} g(q) \geq g(q_0)$, since A is open. Thus, since $g > \pi$ and M is compact, $g \geq \pi + \alpha$. Now choose $\delta > 0$ so small that the length ℓ of the spherical image of E_δ is $\leq \pi + \alpha$ (this is possible since $\ell \rightarrow \pi$ as $\delta \rightarrow 0$). Next, for $(q, t) \in M \times \mathbf{S}^1$, let $\tilde{G}_F(q, t)$ be the normalized projection of $G_F(q, t)$ into $df(T_q M)^\perp$, i.e.,

$$\tilde{G}_F(q, t) := \frac{\sum_{i=1}^2 \langle G_F(q, t), N_i(q) \rangle N_i(q)}{\sqrt{\sum_{i=1}^2 \langle G_F(q, t), N_i(q) \rangle^2}}.$$

Also, for fixed $t \in \mathbf{S}^1$, let $F_t(q) := F(q, t)$. Then, by the tubular neighborhood theorem, $F_t: M \rightarrow \mathbf{R}^{n+1}$ is a smooth immersion for small ε . Further, as $\varepsilon \rightarrow 0$, F_t converges to f with respect to the \mathcal{C}^1 -topology. Thus, for each $q \in M$, the normal

plane $dF_t(T_q M)^\perp$ (which contains $G_F(q, t)$) converges to $df(T_q M)^\perp$. Consequently G_F is well-defined for small ε , and converges to \tilde{G}_F as $\varepsilon \rightarrow 0$. So it suffices to check that $\tilde{G}_F(M \times \mathbf{S}^1) \subset A$, which follows from our choice of δ . Indeed for each fixed $q \in M$, $\tilde{G}_F(\{q\} \times \mathbf{S}^1)$ is the spherical image of the figure eight curve $\sum_{i=1}^2 E_\delta^i(t) N_i(q)$ in $df(T_q M)^\perp$, which is an arc of C_q with midpoint $N_1(q)$ and length $\leq \pi + \alpha$. \square

4. NOTES

4.1. It is well-known that $G_f(M) = \mathbf{S}^n$ for any embedding $f: M^n \rightarrow \mathbf{R}^{n+1}$ of a closed oriented n -manifold [7, p. 187]. More generally, this also holds for ‘‘Alexandrov embeddings’’, i.e., immersions $f: M \rightarrow \mathbf{R}^{n+1}$ which may be extended to an immersion $\bar{f}: \bar{M} \rightarrow \mathbf{R}^{n+1}$ of a compact $(n+1)$ -manifold \bar{M} with $\partial \bar{M} = M$. Indeed if v is any vector field along M which points ‘‘outward’’ with respect to \bar{M} , then for $p \in M$, the normalized projection of $df(v(p))$ into the line $df(T_p M)^\perp$ defines a normal vector field $M \rightarrow \mathbf{S}^n$ which coincides with G_f (after a reflection of G_f if necessary). Then, for any $u \in \mathbf{S}^n$, if p is a point which maximizes the height function $\langle \cdot, u \rangle$ on M , we have $G_f(p) = u$. On the other hand, being only regularly homotopic to an embedding, is not enough to ensure that $G_f(M) = \mathbf{S}^n$. Indeed the example in Figure 2 is regularly homotopic to an embedded torus of revolution [12].

4.2. If $G_f(M) \neq \mathbf{S}^n$ for an immersion $f: M^n \rightarrow \mathbf{R}^{n+1}$ of an oriented n -manifold, then, as is well-known [11], M must be parallelizable. Here we include a brief geometric argument for this fact. If $(0, \dots, 0, 1) \notin G_f(M)$, we may define a continuous map $F: TM \rightarrow \mathbf{R}^n \simeq \mathbf{R}^n \times \{0\} \subset \mathbf{R}^{n+1}$ as follows, c.f. [5, Lemma 2.2]. There is a continuous map $\mathbf{S}^n \setminus \{(0, \dots, 0, 1)\} \xrightarrow{\rho} SO(n+1)$, $u \mapsto \rho_u$ such that $\rho_u(u) = (0, \dots, 0, -1)$. Let $\pi: TM \rightarrow M$ be the canonical projection, and for $X \in TM$ set $F(X) := \rho_{G_f(\pi(X))}(df(X))$. Also let $F_p := F|_{T_p M}$. Then $\{F_p^{-1}(e_i)\}$, where $\{e_i\}$ is a fixed basis of \mathbf{R}^n , gives a framing for TM as desired. So in particular, when M is closed and $n = 2$, we have $M = \mathbf{T}^2$. The last observation also follows from Gauss-Bonnet theorem via degree theory when f is \mathcal{C}^2 ; since if $G_f(M) \neq \mathbf{S}^2$, then

$$0 = \deg(G_f) = \frac{1}{4\pi} \int_M \det(dG_f) = \frac{1}{4\pi} \int_M K = \frac{1}{2} \chi(M),$$

where K is the Gaussian curvature and χ is the Euler characteristic.

4.3. To generate some concrete examples of the immersions $f: M^{n-1} \rightarrow \mathbf{R}^n \simeq \mathbf{R}^n \times \{0\}$ in Theorem 1.3, note that if $f_0: M_0^{n-k-1} \rightarrow \mathbf{R}^{n-k} \times \{0\}$ is any immersion such that $f_0(M_0)$ is disjoint from the subspace $L := \mathbf{R}^{n-k-1} \times \{(0, 0)\}$, then spinning f_0 about L yields an immersion $f_1: M_0 \times \mathbf{S}^1 \rightarrow \mathbf{R}^{n-k+1}$ given by

$$f_1(q, t) := \begin{bmatrix} \text{I} & 0 \\ 0 & \begin{array}{cc} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{array} \end{bmatrix} \begin{bmatrix} f_0^1(q) \\ \vdots \\ f_0^{n-k}(q) \\ 0 \end{bmatrix},$$

where f_0^i are the components of f_0 . Thus, for instance, one may inductively construct immersions of $\mathbf{S}^{n-k-1} \times \mathbf{T}^k$ in \mathbf{R}^n , for $k = 1, \dots, n-1$. More generally, if $M^{n-1} \times \mathbf{S}^1$ is parallelizable, then so is the open manifold $M^{n-1} \times (0, 1)$, which may be immersed in \mathbf{R}^n [10] by the h -principle [7], or more specifically, the “holonomic approximation theorem” of Eliashberg-Mishachev [1, 5].

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