H-PRINCIPLES FOR HYPERSURFACES WITH PRESCRIBED PRINCIPLE CURVATURES AND DIRECTIONS

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Abstract. We prove that any compact orientable hypersurface with boundary immersed (resp. embedded) in Euclidean space is regularly homotopic (resp. isotopic) to a hypersurface with principal directions which may have any prescribed homotopy type, and principal curvatures each of which may be prescribed to within an arbitrary small error of any constant. Further we construct regular homotopies (resp. isotopies) which control the principal curvatures and directions of hypersurfaces in a variety of ways. These results, which we prove by holonomic approximation, establish some h-principles in the sense of Gromov, and generalize theorems of Gluck and Pan on embedding and knotting of positively curved surfaces in 3-space.

1. Introduction

In [7] Gluck and Pan used explicit constructions to prove that any compact orientable surface with boundary in $\mathbf{R}^3$ is regularly homotopic to one with positive Gaussian curvature. Here we generalize that result to hypersurfaces with prescribed signs of principal curvatures in $\mathbf{R}^{n+1}$. Further we show that the principal directions of these hypersurfaces may be prescribed up to homotopy as well. Our proofs are based on holonomic approximation [2], which is one of the main tools for establishing h-principles in the sense of Gromov [5, 9].

Throughout this paper, M is a smooth (C^∞) oriented compact connected n dimensional manifold with nonempty boundary. An *immersion* $f\colon M \to \mathbf{R}^{n+1}$ is a C^1 mapping such that $\mathrm{rank}(df_p) = n$ for all $p \in M$, where df_p denotes the differential map of f at p. If, furthermore, f is one-to-one, we say that it is an *embedding*. Two immersions $f_0, f_1\colon M \to \mathbf{R}^{n+1}$ are *regularly homotopic* provided there exists a continuous family of immersions $f_t\colon M \to \mathbf{R}^{n+1}$, $t \in [0,1]$, such that $t \mapsto d(f_t)_p$ is also continuous, for all $p \in M$. If f_t is an embedding for all $t \in [0,1]$, then we say that f_0 and f_1 are *isotopic*. By a *frame* on M we mean a set $E := \{E_1, \dots, E_n\}$ of independent vector fields $E_i\colon M \to TM$ which is consistent with the orientation of M. By Lemma 2.2 below, if there exists an immersion

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$M \to \mathbf{R}^{n+1}$, then M admits a frame (i.e. it is *parallelizable*). Two frames E^1 and E^2 are said to be *homotopic* provided that there exists a continuous family of frames E^t, $t \in [0,1]$, joining E^0 and E^1.

Theorem 1.1. *Let $f\colon M \to \mathbf{R}^{n+1}$ be an immersion. For any $\varepsilon > 0$, constants $\lambda_i \in \mathbf{R}$, $i = 1, \dots, n$, and frame E on M, there exists a regular homotopy $f_t\colon M \to \mathbf{R}^{n+1}$, $t \in [0,1]$, such that $f_0 = f$, f_1 is C^∞, the principal directions of f_1 define a frame which is homotopic to E, and, for all $p \in M$,*

$$|k_i^{f_1}(p) - \lambda_i| < \varepsilon,$$

where $k_i^{f_1}$ are the principal curvatures of f_1. Furthermore, if f is an embedding, then f_t is an isotopy; in particular, f_1 is embedded as well.

In particular, any homotopy class of frames on M occurs as the principal directions of an immersion $M \to \mathbf{R}^{n+1}$, whose principal curvatures have prescribed signs. Moreover, Theorem 1.1 indicates that the homotopy classes of principal frames on M are unrelated to the regular homotopy classes of immersions $M \to \mathbf{R}^{n+1}$. Note that, due to integrability conditions given by the Gauss and Codazzi-Mainardi equations, it is not possible to prescribe arbitrarily the exact values of the principal curvatures. For instance, the only surfaces in 3-space with constant principal curvatures are the sphere, cylinder, and the plane (in which case either one of the principal curvatures is zero, or they are all equal). Thus the above inequality is the best possible result in prescribing principal curvatures as arbitrary constants. On the other hand, if we forgo embeddedness, it is not difficult to show that equality can be achieved whenever the integrability conditions have a local solution:

Theorem 1.2. *Let $f\colon M \to \mathbf{R}^{n+1}$ be an immersion. For any embedded disk $\Sigma \subset \mathbf{R}^{n+1}$, there exists a regular homotopy $f_t\colon M \to \mathbf{R}^{n+1}$, $t \in [0,1]$, such that $f_0 = f$, and $f_1(M) \subset \Sigma$. In particular, if Σ has constant principal curvatures λ_i, $i = 1, \dots, n$, then, for all $p \in M$,*

$$k_i^{f_1}(p) = \lambda_i.$$

Furthermore, for any constant $C \in \mathbf{R}$, there exists f_t such that, for all $p \in M$,

$$K^{f_1}(p) = C,$$

where K^{f_1} is the Gaussian curvature of f_1.

In the case of $n = 2$, and zero Gauss curvature, the above theorem has been proved by Peter Røgen [16]. Indeed, he showed that any compact embedded surface with boundary in $\mathbf{R}^3$ is isotopic to a flat surface. It is reasonable to conjecture that, in the case of Gaussian curvature in the above theorem, if f is an embedding, then there exists f_t which is actually an isotopy. Further, we should mention that using Monge-Ampère equations, Guan and Spruck [11], as well as Trudinger and Wang [17], have shown that if a closed submanifold bounds a positively curved hypersurface, then it also bounds a hypersurface with constant curvature.

Using direct methods, Gluck and Pan also showed that any pair of compact surfaces with boundary and positive Gaussian curvature immersed in $\mathbf{R}^3$, which are regularly homotopic, are homotopic through a family of positively curved surfaces. Using the parametric version of the holonomic approximation theorem, we generalize their result as follows:

Theorem 1.3. *Let $f_0, f_1 \colon M \to \mathbf{R}^{n+1}$ be C^2 immersions, and suppose that there exists a regular homotopy $f_t \colon M \to \mathbf{R}^{n+1}$, $t \in [0,1]$.*

(i) *If the principal curvatures of f_0 and f_1 are all positive (or all negative), then there exists a regular homotopy $\overline{f}_t \colon M \to \mathbf{R}^{n+1}$ with $\overline{f}_0 = f_0$, $\overline{f}_1 = f_1$, and principal curvatures which are all positive (or all negative).*

(ii) *If f_0 and f_1 have constant Gaussian curvatures C_0 and C_1, respectively, then, for any continuous function C_t, and $\varepsilon > 0$, there exists $\overline{f}_t$ with*

$$|K^{\overline{f}_t}(p) - C_t| < \varepsilon,$$

for all $p \in M$, and $t \in [0,1]$.

(iii) *If f_0 and f_1 have positive (resp. negative) Gaussian curvature, then there exists $\overline{f}_t$ with positive (resp. negative) Gaussian curvature.*

(iv) *If the principal curvatures of f_0 and f_1 are constant, and their corresponding principal directions define homotopic frames, then, for any $\epsilon > 0$, and continuous functions $\lambda_i^t \colon M \to \mathbf{R}$, with $\lambda_i^0 = k_i^{f_0}$, and $\lambda_i^1 = k_i^{f_1}$, there exists $\overline{f}_t$ such that*

$$|k_i^{\overline{f}_t}(p) - \lambda_i^t| < \varepsilon,$$

for all $p \in M$, and $t \in [0,1]$.

(v) *If the principal curvatures $k_i^{f_0}$ and $k_i^{f_1}$ have the same sign, and their corresponding principal directions define homotopic frames; then there exists $\overline{f}_t$ which preserves the sign of each principal curvature.*

In all the above cases, whenever f_t is an isotopy*, then $\overline{f}_t$ is an isotopy as well.*

Papers of Gluck and Pan, Guan and Spruck, and Trundinger and Wang, which we mentioned above, are parts of a recent wave of interest in studying locally convex hypersurfaces with boundary in Euclidean space. For other recent results in this area see [1, 6, 10, 11, 12, 16]. The study of regular homotopy subject to curvature constraints goes back to E. Feldman [3, 4] who studied regular homotopy classes of submanifolds with nonvanishing mean curvature. See also the papers of J. Little [13, 14, 15] for other results on space curves.

The above theorems will be proved in Section 3, after we establish some basic lemmas, and review the prerequisites concerning jet bundles and holonomy.

2. Preliminaries

As mentioned in the introduction, throughout this paper M is a smooth oriented compact connected n dimensional manifold with nonempty boundary. By a *family* of mappings we mean a continuous family. The parameters are often denoted by t or s, which, unless stated otherwise, range from 0 to 1. By a *retraction* we mean a family of smooth embeddings $r_t\colon M \to M$, such that $r_0 = id_M$, i.e., $r_0(p) = p$ for all $p \in M$. A *subpolyhedron* $P \subset M$, is a subcomplex of a triangulation of M. We say that P is *proper*, provided that no simplex of P has dimension n. The following is a well-known topological fact [2, p. 40] which we use often.

Lemma 2.1 (Existence of a Core)**.** *There exists a proper subpolyhedron $P \subset M$ such that for any open neighborhood U of P, there exists a retraction $r_t\colon M \to M$ with $r_1(M) \subset U$.*

The subpolyhedron whose existence is given by the above lemma is known as a *core* of M. The following observation, which has an elementary proof, will also be used throughout the paper.

Lemma 2.2 (Existence of a Frame)**.** *If there exists an immersion $f\colon M \to \mathbf{R}^{n+1}$, then M is parallelizable.*

Proof. It is enough to show that there exists a continuous and fiberwise nonsingular linear map from the tangent bundle TM to $\mathbf{R}^n$. The pull back of this map, applied to the standard basis $e_1, \dots, e_n$ of $\mathbf{R}^n$, will then yield a frame on M. To obtain this map, consider the differential map of f at p

$$df_p\colon T_pM \to T_{f(p)}\mathbf{R}^{n+1} \simeq \mathbf{R}^{n+1}.$$

For each $p \in M$ we will define a proper rotation $\rho_p \in SO_{n+1}$ which maps $df_p(T_pM)$ to $\mathbf{R}^n \times \{0\} \subset \mathbf{R}^{n+1}$, as follows.

First note that, since M is orientable by assumption, there exists a continuous mapping (the Gauss map) $N^f\colon M \to \mathbf{S}^n$ such that $N^f(p)$ is orthogonal to $df_p(T_pM)$ for all $p \in M$. Let $P \subset M$ be a core (Lemma 2.1). After a convolution of f, we may assume that f is smooth. In particular, N^f will be C^1. Now recall that, by definition, P has (n dimensional Lebesque) measure zero in M. Thus it follows that $N^f(P)$ also has measure zero in $\mathbf{S}^n$ [2, 2.3.1]. In particular, $N^f(P)$ does not cover $\mathbf{S}^n$. So, after a rotation, and composing f with a retraction of M into a neighborhood of P, we may assume that, for all $p \in M$,

$$N^f(p) \neq -e_{n+1} := -(0, \dots, 0, 1).$$

If $N^f(p) = e_{n+1}$, then $df_p(T_pM)$ coincides with $\mathbf{R}^n \times \{0\} \subset \mathbf{R}^{n+1}$, and we define ρ_p to be the identity element in SO_{n+1}. Otherwise, there exists a unique 2 dimensional plane Π_p, spanned by $N^f(p)$ and e^{n+1}. We may then define ρ_p as the element in SO_{n+1} which fixes the orthogonal complement of Π_p in $\mathbf{R}^{n+1}$, rotates Π_p by an angle which is less than π, and maps $N^f(p)$ to e_{n+1}. It is clear that when $N^f(p)$ is close to e_{n+1}, ρ_p is close to the identity. Thus $p \mapsto \rho_p$ is continuous. Therefore, we obtain a family of mappings

$$\rho_p \circ df_p \colon T_pM \to \mathbf{R}^n \times \{0\} \simeq \mathbf{R}^n.$$

Since this is a nonsingular linear map, it follows that the vectors

$$E_i(p) := \Big(\rho_p \circ df_p\Big)^{-1}(e_i),$$

$i = 1, \dots n$, are linearly independent for all $p \in M$. So $\{E_i\}$ is a frame on M. □

Without loss of generality we may henceforth assume that our manifolds are parallelizable, since they all admit an immersion into $\mathbf{R}^{n+1}$. This leads to a trivialization of the corresponding jet bundles, as we describe below.

Let E be a frame on M, g be a Riemannian metric, and $\exp^g \colon TM \to M$ be the corresponding exponential map. For every $p \in M$, we may define a mapping

$$\mathbf{R}^n \ni (x_1, \dots, x_n) \xmapsto{\phi_p^{g,E}} \exp^g \Big(x_1E_1(p) + \cdots + x_nE_n(p)\Big) \in M.$$

Note that $\phi_p^{g,E}(o) = p$, where $o := (0, \dots, 0)$ is the origin of $\mathbf{R}^n$. By the inverse function theorem, $\phi_p^{g,E}$ is a local diffeomorphism at o. Thus $\phi_p^{g,E}$, when restricted to a small neighborhood of the origin, is a local chart for M centered at p, which are called its *normal coordinates* with respect to g and E.

Now, for every $p \in M$, we may define (the matrix representations of) the *first* and *second derivatives* of $f \colon M \to \mathbf{R}^{n+1}$, with respect to (the normal coordinates determined by) g and E, as

$$\begin{aligned} D_{g,E}f(p) &:= \left(\frac{\partial f \circ \phi_p^{g,E}(o)}{\partial x_i}\right)_{1\le i\le n} &&\in \mathcal{M}^{\mathbf{R}}_{(n+1)\times n}\,, \\ D^2_{g,E}f(p) &:= \left(\frac{\partial^2 f \circ \phi_p^{g,E}(o)}{\partial x_i \partial x_j}\right)_{1\le i,j\le n} &&\in \mathcal{SM}^{\mathbf{R}^{n+1}}_{n\times n}\,, \end{aligned}$$

respectively, where $\mathcal{M}^{\mathbf{R}}_{(n+1)\times n}$ is the space of $(n+1)\times n$ matrices with real entries, i.e., the vector space of linear maps $\mathbf{R}^n \to \mathbf{R}^{n+1}$; and, $\mathcal{SM}^{\mathbf{R}^{n+1}}_{n\times n}$ is the space of symmetric $n\times n$ matrices with entries in $\mathbf{R}^{n+1}$, i.e, the vector space of symmetric bilinear maps $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^{n+1}$ (a matrix in $\mathcal{SM}^{\mathbf{R}^{n+1}}_{n\times n}$ may be viewed as a collection $(A_1, \dots, A_{n+1})$ of matrices in $\mathcal{SM}^{\mathbf{R}}_{n\times n}$, then any pair X, $Y \in \mathbf{R}^n$ yields a vector in $\mathbf{R}^{n+1}$ via $(X^TA_1Y, \dots, X^TA_{n+1}Y)$).

Note that

$$D_{g,E}f(p) = \Big(df_p\big(E_i(p)\big)\Big)_{1\le i\le n} =: D_Ef(p),$$

i.e., $D_{g,E}f$ does not depend on g, and hence may be simply denoted by D_Ef. By contrast, $D^2_{g,E}f$ depends on g as well as E.

The jet bundle $J^k(M, \mathbf{R}^{n+1})$ is the space of k-tangency classes $J^k_p(f)$ of C^k functions $f \colon M \to \mathbf{R}^{n+1}$ at $p \in M$; two function f_1, $f_2 \colon M \to \mathbf{R}^{n+1}$ are "k-tangent" at $p \in M$, if their derivatives at p, with respect to some and hence any local chart, are equal up to order k [8]. More formally,

$$J^k(M,\mathbf{R}^{n+1}) := \big\{ J^k_p(f) \mid p\in M,\ f\in C^k(M,\mathbf{R}^{n+1}) \big\},$$

where, for $k = 1, 2$, the tangency classes are defined as

$$J^k_p(f) := \big\{ \varphi \in C^k(M,\mathbf{R}^{n+1}) \mid D^r_{g,E}\varphi(p) = D^r_{g,E}f(p),\ r = 0,\dots,k \big\}.$$

The following observation is immediate:

Lemma 2.3 (Trivialization of Jet Bundles)**.** *Let $f\colon M \to \mathbf{R}^{n+1}$ be a mapping. Each choice of a Riemannian metric g and frame E on M induces the following mappings:*

$$J^1_p(f) \overset{\Theta_E}{\longmapsto} \Big(p, f(p), Df_E(p)\Big), \quad \textit{and} \quad J^2_p(f) \overset{\Theta_{g,E}}{\longmapsto} \Big(p, f(p), D_Ef(p), D^2_{g,E}f(p)\Big),$$

which result in the trivializations of the corresponding jet bundles

$$\begin{aligned} J^1(M,\mathbf{R}^{n+1}) &\simeq M\times\mathbf{R}^{n+1}\times\mathcal{M}^{\mathbf{R}}_{(n+1)\times n},\\ J^2(M,\mathbf{R}^{n+1}) &\simeq M\times\mathbf{R}^{n+1}\times\mathcal{M}^{\mathbf{R}}_{(n+1)\times n}\times\mathcal{SM}^{\mathbf{R}^{n+1}}_{n\times n}, \end{aligned}$$

respectively. □

A section $F\colon M \to J^k(M,\mathbf{R}^{n+1})$, is called *holonomic* provided that there exists a function $f\colon M\to\mathbf{R}^{n+1}$ such that $F(p) = J^k_p(f)$ for all $p\in M$. Thus, for any choice of metric g and frame E on M, F is holonomic, for $k = 1, 2$, if and only if

$$\begin{aligned} \Theta_E\Big(F(p)\Big) &= \Big(p, f(p), D_Ef(p)\Big), \text{ or}\\ \Theta_{g,E}\Big(F(p)\Big) &= \Big(p, f(p), D_Ef(p), D^2_{g,E}f(p)\Big), \end{aligned}$$

respectively, for some function $f\colon M\to\mathbf{R}^{n+1}$.

The following are the holonomic approximation theorems [2] which we require. Since $J^k(M,\mathbf{R}^{n+1})$ is a manifold, it admits a Riemannian metric, which we denote by "dist", and fix for the remainder of the paper. By an *isotopy* $\psi_t\colon M\to M$, we mean a family of diffeomorphisms such that $\psi_0 = id_M$.

Lemma 2.4 (Hol. Approx. Thm.)**.** *Let $F\colon M\to J^k(M,\mathbf{R}^{n+1})$ be a section. For any proper subpolyhedron $P\subset M$, and $\varepsilon>0$, there exists an isotopy $\psi_t\colon M\to M$, an open neighborhood U of $\psi_1(P)$, and a holonomic section $\widetilde{F}\colon U\to J^k(M,\mathbf{R}^{n+1})$ such that* $\operatorname{dist}(F(p),\widetilde{F}(p))<\varepsilon$ *for all $p\in U$.*

One may also assume that the isotopy ψ_t in the above lemma is "δ-small", that is, $\psi_t(p)$ is within a δ-distance of p in M for all p in M and $t\in[0,1]$. However, this feature of the holonomic approximation theorem will not be needed for our results. The next lemma is the parametric version of the previous one:

Lemma 2.5 (Param. Hol. Approx. Thm.)**.** *Let $F_t\colon M\to J^k(M,\mathbf{R}^{n+1})$, be a family of sections. Suppose that F_0 and F_1 are holonomic. For any proper subpolyhedron $P\subset M$, and $\varepsilon>0$, there exists a family of isotopies $\psi^t_s\colon M\to M$, an open neighborhood U_t of $\psi^t_1(P)$, and a family of holonomic sections $\widetilde{F}_t\colon U_t\to$*

$J^k(M, \mathbf{R}^{n+1})$ *such that* $\widetilde{F}_0 = F_0$, $\widetilde{F}_1 = F_1$, *and* $\operatorname{dist}(F_t(p), \widetilde{F}_t(p)) < \varepsilon$ *for all* $p \in U_t$.

Note that each trivialization of the jet bundles has a natural metric given by the product of the Riemannian metric g on M, and the Euclidean distance on the remaining factors in the trivialization. With respect to this metric, and the "dist" metric on $J^k(M, \mathbf{R}^{n+1})$, the mapping $\Theta_{g,E}$ is a homeomorphism. In other words, two jets are close, if, and only if, their images in a trivialization are close:

$$J^k_p(f_1) \approx J^k_p(f_2) \iff \Theta_{g,E}\Big(J^k_p(f_1)\Big) \approx \Theta_{g,E}\Big(J^k_p(f_2)\Big).$$

Since isotopy of a core is a core, it follows, by Lemma 2.1, that if P is a core of M, then there is a family of retractions $r^t_s \colon M \to M$, such that

$$r^t_0 = id_M \quad \text{and} \quad r^t_1(M) \subset U_t,$$

for all t. In particular, if we have a family of mappings $f_t \colon U_t \to \mathbf{R}^{n+1}$, then $f_t \circ r^t_1 \colon M \to \mathbf{R}^{n+1}$ will again be a family (i.e., $t \mapsto f_t \circ r^t_1$ will be continuous).

Let f be an *immersion*, i.e., for all $p \in M$, $\operatorname{rank}(D_E f(p)) = n$. For every $p \in M$, the matrix representation of the *first fundamental form* of $f \colon M \to \mathbf{R}^{n+1}$, with respect to g and E, is given by

$$\mathrm{I}^f_E(p) := D_E f(p)^T \cdot D_E f(p) = \Big(\Big\langle df_p(E_i), df_p(E_j)\Big\rangle\Big)_{1 \le i,j \le n},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product. Note that I^f_E does not depend on g. However, if g^f is the metric which is indued on M by f, i.e.,

$$g^f\Big(E_i(p), E_j(p)\Big) := \Big\langle df_p(E_i), df_p(E_j)\Big\rangle, \tag{1}$$

and we assume that $E(p)$ is orthonormal with respect to g^f, then $\mathrm{I}^f_E(p)$ is the identity matrix, for all $p \in M$.

The *Gauss map* $N^f \colon M \to \mathbf{S}^n \subset \mathbf{R}^{n+1}$ is defined as the point $N^f(p) \in \mathbf{S}^n$, which is orthogonal to the columns of $D_E f(p)$, and such that the augmented matrix $\big(D_E f(p), N^f(p)\big)$ has positive determinant. Note that N^f depends only on the orientation of E. Thus, N^f is uniquely determined, because, by assumption, all frames E have the same orientation determined by M.

The matrix representations of the *second fundamental form*, and the *shape operator* of f at p, with respect to g and E, are given by

$$\mathrm{II}^f_E(p) := \Big\langle D^2_{g,E} f(p), N^f(p)\Big\rangle = -\Big(\Big\langle df_p(E_i), dN^f_p(E_j)\Big\rangle\Big)_{1 \le i,j \le n},$$

$$\mathrm{S}^f_E(p) := \mathrm{II}^f_E(p) \cdot \Big(\mathrm{I}^f_E(p)\Big)^{-1},$$

respectively. Thus neither II^f_E nor S^f_E depends on g. The eigenvalues and determinant of S^f_E define the *principal curvatures*, and the *Gaussian curvature* of f, respectively. Note that every frame E induces an isomorphism

$$T_pM \ni X := x_1 E_1(p) + \cdots + x_n E_n(p) \longmapsto (x_1, \ldots, x_n) =: X_E \in \mathbf{R}^n.$$

We say that a (nonzero) vector $X \in T_pM$ is a *principal direction* of f at p, provided that X_E is an eigenvector of $\mathrm{S}^f_E(p)$. In particular, when $\mathrm{S}^f_E(p)$ is a diagonal matrix, the principal directions of M at p are spanned by $E_i(p)$. Finally note that $\mathrm{II}^f_E(p)$ defines a bilinear form

$$T_pM \times T_pM \ni (X, Y) \overset{\mathrm{II}^f}{\longmapsto} X_E^T \, \mathrm{II}^f_E(p) \, Y_E \in \mathbf{R},$$

which is independent of E. We say that $\mathrm{II}^f_E(p)$ is *positive* (resp. *negative*) *definite*, provided that the corresponding bilinear form is positive (resp. negative) definite. This holds precisely when the principal curvatures of f at p are all positive (resp. negative).

3. Proofs

Following the general philosophy of the h-principle, the central idea of the following arguments is to construct appropriate sections of the jet bundles

$$J^1(M, \mathbf{R}^{n+1}) \quad \text{or} \quad J^2(M, \mathbf{R}^{n+1}),$$

and then approximate them by holonomic sections on a neighborhood of a core of M. Since our manifolds are open, they retract into a neighborhood of the core (Lemma 2.1). In particular, after composing these approximations with a retraction, or a family of retractions, we obtain the desired mappings. Thus the main step of each proof consists of constructing jet bundle sections which satisfy the prescribed conditions in a formal sense. This is facilitated by the parallelizability of our manifolds (Lemma 2.2), which in turn leads to the trivialization of the jet bundles (Lemma 2.3), as described in the previous section.

Proof of Theorem 1.1. Let g^f be the metric which is induced on M by the immersion $f \colon M \to \mathbf{R}^{n+1}$, as defined by (1). After a homotopy of E, via a Gram-Schmidt process, we may assume that E is orthonormal with respect to g^f. The matrix representation of the first fundamental form is then the identity,

$$\mathrm{I}^f_E(p) = Id_{n\times n}$$

for all $p \in M$.

Now, using the trivialization $\Theta_{g^f,E}$ induced by g^f and E on the space of 2-jets (Lemma 2.3), we may define a section $F \colon M \to J^2(M, \mathbf{R}^{n+1})$ by

$$F(p) := \Theta^{-1}_{g^f,E}\Big(p, f(p), D_E f(p), \Lambda\, N^f(p)\Big),$$

where, Λ is a diagonal matrix whose nonzero entries are the prescribed values for principal curvatures,

$$\Lambda := \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix},$$

and thus $\Lambda N^f(p)$ denotes the diagonal matrix whose nonzero entries are $\lambda_i N^f(p)$. In particular, $\Lambda\, N^f(p) \in \mathcal{SM}^{\mathbf{R}^{n+1}}_{n\times n}$. So F is well defined.

Let P be a core of M as in Lemma 2.1, and apply the Holonomic Approximation Theorem (Lemma 2.4) to get the holonomic section $\widetilde{F}\colon U \to \mathbf{R}^{n+1}$, where U is a neighborhood of the perturbed core $\psi_1(P)$. By definition, $\widetilde{F}(p) = J^2_p(\widetilde{f})$ for some function $\widetilde{f}\colon U \subset M \to \mathbf{R}^{n+1}$, and all $p \in U$. Thus, again using the trivialization of the jet bundle, we have

$$\Theta_{g^f,E}\Big(\widetilde{F}(p)\Big) = \Big(p, \widetilde{f}(p), D_E\widetilde{f}(p), D^2_{g^f,E}\widetilde{f}(p)\Big).$$

Since, by Lemma 2.4, $\widetilde{F} \approx F|_U$, it follows that $D_E\widetilde{f} \approx D_E f|_U$. So, since by assumption $D_E f$ is nonsingular on the closure of U, which is compact, it follows that $D_E\widetilde{f}(p)$ is nonsingular, for all $p \in U$. In particular, $\widetilde{f}$ is an immersion. Further, for each $p \in U$,

$$\begin{aligned} \mathrm{S}^{\widetilde{f}}_E(p) &= \Big\langle D^2_{g^f,E}\widetilde{f}(p), N^{\widetilde{f}}(p)\Big\rangle \Big(\mathrm{I}^{\widetilde{f}}_E(p)\Big)^{-1} \\ &\approx \Big\langle \Lambda\, N^f(p), N^f(p)\Big\rangle \Big(\mathrm{I}^f_E(p)\Big)^{-1} \\ &= \Lambda, \end{aligned}$$

where the first equality holds by definition, the middle approximation is due to the fact that $\widetilde{F} \approx F|_U$, and the last equality follows because $\mathrm{I}^f_E(p)$ is the identity matrix.

Thus, for each $p \in U$, the eigenvalues of $\mathrm{S}^{\widetilde{f}}(p)$, i.e., the principal curvatures of $\widetilde{f}$, are close to eigenvalues of Λ which are λ_i by construction. Let $r_t\colon M \to M$ be the retraction into U whose existence is given by Lemma 2.1. Note that, after a perturbation, we may assume that $\widetilde{f}$ is C^∞. Then the immersion $f_1\colon M \to \mathbf{R}^{n+1}$ defined by

$$f_1(p) := \widetilde{f}\big(r_1(p)\big)$$

yields a C^∞ hypersurface whose principal curvatures are as close to λ_i as desired.

Now we show that f is regularly homotopic to f_1. This will be done in two stages. First, set

$$f_t(p) := f\big(r_{2t}(p)\big), \quad 1 \le t \le \frac{1}{2}.$$

Then $f_0 = f$ and $f_{1/2}(M) \subset f(U)$. Note that, since f and $\widetilde{f}$ are C^1 close on U, $f_{1/2} = f \circ r_1$ is C^1-close to $\widetilde{f} \circ r_1 = f_1$ on M. Thus $f_{1/2}$ and f_1 are regularly homotopic, via a linear interpolation:

$$f_t(p) := \big(2 - 2t\big) f_{1/2}(p) + \big(2t - 1\big) f_1(p), \quad \frac{1}{2} \le t \le 1.$$

So f_t is a regular homotopy, joining f and f_1.

Next suppose that f is an embedding. Then it is clear that f_t is an embedding as well for all $t \in [0, 1/2]$. Furthermore, $\widetilde{f}$ will be an embedding, since, by construction, it is C^1 close to f, and M is compact. So f_1 is an embedding, which in turn yields that f_t is an embedding for $t \in [1/2, 1]$, because $f_{1/2}$ and f_1 are C^1 close. Thus we conclude that whenever f is an embedding, f_t is an embedding as well for all t.

It remains to show that a set of principal directions of f^1,

$$\mathcal{P}^{f_1} := \left\{ \mathcal{P}^{f_1}_1, \ldots, \mathcal{P}^{f_1}_n \right\},$$

forms a frame on M which is homotopic to the prescribed frame $E := \{E_1, \ldots, E_n\}$. First recall that a homotopy of two frames is a family of frames joining them. Secondly note that, after a perturbation of λ_i we may assume that no two of them are equal, and repeat the above argument. Now choosing ε sufficiently small, we can assume that no two of the principal curvatures of f_1 are equal. Then the principal directions of f_1 are independent at each point. So they may be used to define a frame $\mathcal{P}^{f_1}$ on M.

To show that $\mathcal{P}^{f_1} \simeq E$, i.e., $\mathcal{P}^{f_1}$ is homotopic to E, we use the following terminology. For any frame $\mathcal{F}$ on M, let $\mathcal{F}|_{r_t(M)}$ denote the frame on M given by the inverse of the differential map $d(r_t)$, applied to the restriction of $\mathcal{F}$ on $r_t(M)$,i.e., for all $p \in M$,

$$\mathcal{F}\Big|_{r_t(M)}(p) := \Big(d(r_t)_p\Big)^{-1}\Big(\mathcal{F}\big(r_t(p)\big)\Big).$$

Note that $\mathcal{F}|_{r_t(M)}$ gives a homotopy between $\mathcal{F}$ and $\mathcal{F}|_{r_1(M)}$. So we may write

$$\mathcal{F}\Big|_{r_1(M)} \simeq \mathcal{F}.$$

In other words, *any frame on M is homotopic to the pull back of its restriction to a neighborhood of any core.*

So, to prove that $E \simeq \mathcal{P}^{f_1}$, we need only to check that $E|_{r_1(M)} \simeq \mathcal{P}^{f_1}|_{r_1(M)}$. Since, for all $p \in U$, the eigenvectors of $\mathrm{S}^{\widetilde{f}}_E(p)$ are close to those of Λ, and Λ is diagonal, it follows that, for $i = 1, \ldots, n$, we may choose a principal direction $\mathcal{P}^{\widetilde{f}}_i(p)$ arbitrarily close to $E_i(p)$. Then a linear interpolation yields that $E|_{r_1(M)} \simeq \mathcal{P}^{\widetilde{f}}|_{r_1(M)}$. So it remains to show that $\mathcal{P}^{\widetilde{f}}|_{r_1(M)} \simeq \mathcal{P}^{f_1}|_{r_1(M)}$.

To see the latter claim, note that we may assume without loss of generality that $r_1(p) = p$ for all p in $U' \subset U$, where U' is a neighborhood of the core. Then $\mathcal{P}^{\widetilde{f}}|_{r_1(M)}$ and $\mathcal{P}^{f_1}|_{r_1(M)}$ coincide on U'. Therefore, as we had argued above, since M retracts into U', it follows that $\mathcal{P}^{\widetilde{f}}|_{r_1(M)} \simeq \mathcal{P}^{f_1}|_{r_1(M)}$. Thus we conclude that $E \simeq \mathcal{P}^{f_1}$. □

The next proof uses the hyperplane rotation technique of Lemma 2.2, and both versions of the holonomic approximation theorem.

Proof of Theorem 1.2. Fix a Riemannian metric g and frame E on M. As in the proof of Lemma 2.2, after retracting M into a neighborhood of a core, and a rotation, we may assume that

$$N^f(p) \neq -e_{n+1}$$

for all $p \in M$. If, furthermore, $N^f(p) \neq e_{n+1}$, let $\Pi_p \subset \mathbf{R}^{n+1}$ be as in the proof of Lemma 2.2, i.e., the plane determined by $N^f(p)$ and e_{n+1}.

Since $N^f(p) \neq -e_{n+1}$, there is a unique geodesic in $\mathbf{S}^n$ which connects these two points. Note that this geodesic lies in Π_p. Let $\gamma_p\colon [0,1] \to \mathbf{S}^n$ be a constant speed parametrization of this geodesic such that $\gamma_p(0) = N^f(p)$, and $\gamma_p(1) = e_{n+1}$. Now, for each $p \in M$, we may define a continuous curve

$$[0,1] \ni t \longmapsto \rho_p^t \in SO_{n+1},$$

by requiring that ρ_p^t be the (unique) element which fixes the orthogonal complement of Π_p in $\mathbf{R}^{n+1}$, rotates Π_p by less than π, and maps $N^f(p)$ to $\gamma_p(t)$. If $N^f(p) = e_{n+1}$, we define ρ_p^t to be the identity element in SO_{n+1} for all t. Thus we obtain a family of curves ρ_p^t defined on all of M.

Define a section $F\colon M \to J^1(M, \mathbf{R}^{n+1})$ by

$$F(p) := \Theta_E^{-1}\Big(\, p, f(p), \rho_p^1\big(D_E f(p)\big)\,\Big).$$

Apply Lemma 2.4 to obtain the holonomic section $\widetilde{F}\colon U \to J^1(M, \mathbf{R}^{n+1})$. Then

$$\Theta_E\Big(\widetilde{F}(p)\Big) = \Big(p, \widetilde{f}(p), D_E\widetilde{f}(p)\Big).$$

Since $D_E\widetilde{f}(p) \approx \rho_p^1(Df(p))$, it follows $N^{\widetilde{f}}(p) \approx \rho_p^1\big(N^f(p)\big) = e_{n+1}$, for all p in a neighborhood U of a core of M. In particular, the tangent hyperplanes to $\widetilde{f}(U)$ are not "vertical". So if, for $t \in [0,1]$, we define $\pi_t\colon \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ by

$$\pi_t(x_1, \ldots, x_{n+1}) := \big(x_1, \ldots, x_n, (1-t)x_{n+1}\big),$$

then $\pi_t \circ \widetilde{f}$ is a regular homotopy, which sends $\widetilde{f}(U)$ to $\mathbf{R}^n \times \{0\}$. So, since U has compact closure (because M is compact), and Σ is locally a graph over $\mathbf{R}^n$ (e.g., via the inverse of a projection into a tangent space), it follows that there exists a regular homotopy $\widetilde{f}_t\colon U \to \mathbf{R}^{n+1}$, with $\widetilde{f}_0 = \widetilde{f}$ and $\widetilde{f}_1(U) \subset \Sigma$.

Now recall that, by Lemma 2.1, there exists a retraction $r_t\colon M \to M$, with $r_0 = id_M$ and $r_1(M) \subset U$. Note that $f \circ r_t\colon M \to \mathbf{R}^{n+1}$ is a regular homotopy between f and $f \circ r_1$. Thus to prove that f admits a regular homotopy into Σ, it suffices to show that $f \circ r_1$ is regularly homotopic to $\widetilde{f} \circ r_1$. Composing this homotopy with $\widetilde{f}_t \circ r_1$ would then yield the desired homotopy of f into a mapping whose image lies in Σ.

To see that $f \circ r_1$ is regularly homotopic to $\widetilde{f} \circ r_1$, it is convenient to adopt the following convention for the rest of the proof. We identify $r_1(M)$ with M, and write f and $\widetilde{f}$ instead of $f \circ r_1$ and $\widetilde{f} \circ r_1$ respectively . Let $\overline{\mathcal{M}}^{\mathbf{R}}_{(n+1)\times n}$ be the space of $(n+1) \times n$ matrices of full rank n, and define $A_t\colon M \to \overline{\mathcal{M}}^{\mathbf{R}}_{(n+1)\times n}$, by

$$A_t(p) := \begin{cases} \rho_p^{2t}\big(D_E f(p)\big), & \text{if } 0 \leq t \leq \frac{1}{2} \\ (2-2t)\rho_p^1\big(D_E f(p)\big) + (2t-1)D_E\widetilde{f}(p), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Note that A_t is well defined, i.e., $A_t(p)$ has full rank for all $p \in M$, because Df and $D\widetilde{f}$ both have full rank and are arbitrarily close on U.

Now define a family of sections of $J^1(M, \mathbf{R}^{n+1})$ by

$$F_t(p) := \Theta_E^{-1}\Big(p, (1-t)f(p) + t\widetilde{f}(p), A_t(p)\Big).$$

Then, F_0 and F_1 are holonomic. So, applying Lemma 2.5, we obtain a family of holonomic sections $\overline{F}_t\colon U_t \to J^1(M, \mathbf{R}^{n+1})$. This in turn yields a family of functions $\overline{f}_t\colon U_t \to \mathbf{R}^{n+1}$, given by

$$\Theta_E\Big(\overline{F}_t(p)\Big) = \Big(p, \overline{f}_t, D_E\overline{f}_t(p)\Big).$$

Then $\overline{f}_t$ gives a homotopy between f and $\widetilde{f}$, after appropriate compositions with retractions which map M into U_t.

More explicitly, recall that, as we mentioned in Section 2, there exists a family of retractions $r_s^t\colon M \to M$ with $r_0^t = id_M$, and $r_1^t(M) \subset U_t$. These yield a 3 part homotopy, $f_t\colon M \to \mathbf{R}^{n+1}$ given by

$$f_t := \begin{cases} f \circ r^0_{3t}, & \text{if } 0 \le t \le \frac{1}{3}, \\ \overline{f}_{3t-1} \circ r_1^{3t-1}, & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \\ \widetilde{f} \circ r^1_{(-3t+3)}, & \text{if } \frac{2}{3} \le t \le 1. \end{cases}$$

Thus f_t connects f to $\widetilde{f}$ as desired. □

Proof of the first part of Theorem 1.3 rests on the following fact, which makes it possible to apply the holonomic approximation theorems. Recall that, as we mentioned in Section 2, a matrix $A \in \mathcal{SM}^{\mathbf{R}}_{n\times n}$ is positive (resp. negative) definite, provided that the associated bilinear form $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is positive (resp. negative) definite; that is, $X^TAX > 0$ (resp. < 0), for all nonzero vectors $X \in \mathbf{R}^n$. Thus if A and B are positive (resp. negative) definite matrices, then so is $(1-t)A + tB$ for all $t \in [0,1]$. Hence we record:

Lemma 3.1. *In the space of symmetric matrices $\mathcal{SM}^{\mathbf{R}}_{n\times n}$, the subset of positive definite matrices, and negative definite matrices, are each convex.* □

In particular, the space of positive definite and negative definite matrices is each path connected.

Proof of Theorem 1.3. Each of the five enumerated parts of the theorem will be proved in the corresponding section below. The main distinguishing feature among them is in the construction of a section of $J^2(M, \mathbf{R}^{n+1})$, which satisfies the given conditions in a formal sense. This will become more involved in parts (iv) and (v), where we will impose a family of metrics and frames on M. In the first three parts, however, we assume that M is endowed with a fixed Riemannian metric g and frame E.

(i) Define $A_t\colon M \to \mathcal{SM}_{n\times n}^{\mathbf{R}^{n+1}}$ by

$$A_t(p) := \Big(D^2_{g,E} f_t(p)\Big)^{\top} + \Big((1-t)\,\mathrm{II}_E^{f_0}(p) + t\,\mathrm{II}_E^{f_1}(p)\Big) N^{f_t}(p),$$

where $\big(D^2_{g,E} f_t(p)\big)^{\top}$ is the tangential component of $D^2_{g,E} f_t(p)$, i.e.,

$$\begin{aligned} \Big(D^2_{g,E} f_t(p)\Big)^{\top} &:= D^2_{g,E} f_t(p) - \Big\langle D^2_{g,E} f_t(p), N^{f_t}(p)\Big\rangle N^{f_t}(p) \\ &= D^2_{g,E} f_t(p) - \mathrm{II}_E^{f_t}(p) N^{f_t}(p). \end{aligned}$$

Let the sections $F_t\colon M \to J^2(M, \mathbf{R}^{n+1})$ be given by

$$F_t(p) := \Theta_{g,E}^{-1}\Big(\, p, f_t(p), D_E f_t(p), A_t(p)\Big).$$

Note that, for $s = 0$, 1,

$$A_s(p) = \Big(D^2_{g,E} f_s(p)\Big)^{\top} + \mathrm{II}_E^{f_s}(p) = D^2_{g,E} f_s(p).$$

Thus F_0 and F_1 are holonomic. Now let P be a core of M as in Lemma 2.1, and apply Lemma 2.5 to get the holonomic sections

$$\Theta_{g,E}\Big(\widetilde{F}_t(p)\Big) := \Big(\, p, \widetilde{f}_t(p), D_E \widetilde{f}_t(p), D^2_{g,E}\widetilde{f}_t(p)\Big).$$

Since $\widetilde{F}_t \approx F_t$ on a neighborhood U_t of the perturbed core $\psi_1^t(P)$, it follows that, for all $p \in U_t$,

$$\begin{aligned} \mathrm{II}_E^{\widetilde{f}_t}(p) &= \Big\langle D^2_{g,E}\widetilde{f}_t(p), N^{\widetilde{f}_t}(p)\Big\rangle \\ &\approx \Big\langle A_t(p), N^{f_t}(p)\Big\rangle \\ &= (1-t)\,\mathrm{II}_E^{f_0}(p) + t\ \mathrm{II}_E^{f_1}(p). \end{aligned}$$

Now recall that, as mentioned at the end of Section 2, since the principal curvatures of f_0 and f_1 are all positive (resp. negative) by assumption, it follows that $\mathrm{II}_E^{f_0}(p)$ and $\mathrm{II}_E^{f_1}(p)$ are positive (resp. negative) definite matrices. But the space of positive (resp. negative) definite matrices is convex (Lemma 3.1). So the above computation shows that, for all t, $\mathrm{II}_E^{\widetilde{f}_t}(p)$ is positive (resp. negative) definite. This in turn yields that $\widetilde{f}_t$ has positive (resp. negative) principal curvatures on U_t for all t.

Finally, similar to the end of the proof of Theorem 1.2, we recall that there exists a family of retractions $r_s^t\colon M \to M$ with $r_0^t = id_M$, and $r_1^t(M) \subset U_t$. Using these retractions, we define our desired homotopy $\overline{f}_t\colon M \to \mathbf{R}^{n+1}$ in three parts

$$\overline{f}_t := \begin{cases} f_0 \circ r^0_{3t}, & \text{if } 0 \le t \le \frac{1}{3}, \\ \widetilde{f}_{3t-1} \circ r_1^{3t-1}, & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \\ f_1 \circ r^1_{(-3t+3)}, & \text{if } \frac{2}{3} \le t \le 1. \end{cases} \tag{2}$$

Recall that $\widetilde{F}_0 = F_0$ on U_0, and $\widetilde{F}_1 = F_1$ on U_1. In particular $f_0|_{U_0} = \widetilde{f}_0|_{U_0}$, and $\widetilde{f}_1|_{U_1} = f_1|_{U_1}$. Thus $\overline{f}_t$ is well defined. Also note that if f_t is an isotopy, then we may assume that $\widetilde{f}_t$ is an isotopy as well; because, for every $t \in [0,1]$, f_t and $\widetilde{f}_t$ are C^1-close by construction, embeddings are open with respect to the C^1-norm in the space of immersions, and $[0,1]$ is compact. Thus whenever f_t is an isotopy, we can make sure that $\overline{f}_t$ is an isotopy as well, by choosing $\widetilde{f}_t$ sufficiently C^1-close to f_t.

(ii) Let

$$\alpha(t) := C_t \, \frac{\det\Big(\mathrm{I}_E^{f_t}(p)\Big)}{\det\Big((1-t)\,\mathrm{II}_E^{f_0}(p) + t\,\mathrm{II}_E^{f_1}(p)\Big)},$$

and repeat part (i) with

$$A_t(p) := \Big(D^2_{g,E} f_t(p)\Big)^{\top} + \sqrt[n]{\alpha(t)}\Big((1-t)\,\mathrm{II}_E^{f_0}(p) + t\,\mathrm{II}_E^{f_1}(p)\Big) N^{f_t}(p).$$

Note that, for $s = 0$, 1,

$$\alpha(s) = C_s \frac{\det\Big(\mathrm{I}_E^{f_s}(p)\Big)}{\det\Big(\mathrm{II}_E^{f_s}(p)\Big)} = \frac{C_s}{\det\big(\mathrm{S}^{f_s}(p)\big)} = 1.$$

Thus, as in part (i), F_0 and F_1 are holonomic. Further, similar to part (i), we have

$$\begin{aligned}
\mathrm{II}_E^{\widetilde{f}_t}(p) &= \Big\langle D^2_{g,E}\widetilde{f}_t(p), N^{\widetilde{f}_t}(p)\Big\rangle \\
&\approx \Big\langle A_t(p), N^{f_t}(p)\Big\rangle \\
&= \sqrt[n]{\alpha(t)}\Big((1-t)\,\mathrm{II}_E^{f_0}(p) + t\,\mathrm{II}_E^{f_1}(p)\Big).
\end{aligned}$$

In particular, for all $p \in U_t$,

$$\begin{aligned}
K^{\widetilde{f}_t}(p) &= \frac{\det\Big(\mathrm{II}_E^{\widetilde{f}_t}(p)\Big)}{\det\Big(\mathrm{I}_E^{\widetilde{f}_t}(p)\Big)} \\
&\approx \alpha(t)\,\frac{\det\Big((1-t)\,\mathrm{II}_E^{f_0}(p) + t\,\mathrm{II}_E^{f_1}(p)\Big)}{\det\Big(\mathrm{I}_E^{f_t}(p)\Big)} \\
&= C_t,
\end{aligned}$$

as desired.

(iii) Repeat part (ii) with

$$C_t(p) := (1-t)K^{f_0}(p) + tK^{f_1}(p).$$

(iv) Let E^0 and E^1 be frames defined by the principal directions of f_0 and f_1 respectively, and E^t be a homotopy joining them. Let $g^t := g^{f_t}$ be the metric induced on M by f^t. After a Gram-Schmidt process, we may assume that E^t is orthonormal with respect to g^t. In particular,

$$\mathrm{I}^{f_t}_{E^t}(p) = Id_{n\times n},$$

for all $p \in M$ and all t. Define $A_t \colon M \to \mathcal{SM}^{\mathbf{R}^{n+1}}_{n\times n}$ by

$$A_t(p) := \Big(D^2_{g^t,E^t} f_t(p)\Big)^{\top} + \Lambda^t\, N^{f_t}(p),$$

where

$$\Lambda^t := \begin{pmatrix} \lambda^t_1 & & 0 \\ & \ddots & \\ 0 & & \lambda^t_n \end{pmatrix}.$$

Since for $s = 0,\, 1$, E^s is the frame of principal directions of f_s, and $\mathrm{I}^{f_s}_{E^s}(p)$ is the identity matrix,

$$\mathrm{II}^{f_s}_{E^s}(p) = \mathrm{S}^{f_s}_{E^s}(p)\Big(\mathrm{I}^{f_s}_{E^s}(p)\Big)^{-1} = \mathrm{S}^{f_s}_{E^s}(p) = \Lambda^s.$$

Consequently, for $s = 0,\, 1$,

$$A_s(p) = D^2_{g^s,E^s} f_s(p) - \mathrm{II}^{f_s}_{E^s}(p) N^{f_s}(p) + \Lambda^s N^{f_s}(p) = D^2 f_{g^s,E^s}(p).$$

So, if we define the sections $F_t \colon M \to J^2(M, \mathbf{R}^{n+1})$ by

$$F_t(p) := \Theta^{-1}_{g_t,E^t}\Big(p, f_t(p), D_{E^t} f_t(p), A_t(p)\Big),$$

then F_0 and F_1 are holonomic. Now let P be a core of M as in Lemma 2.1, and apply Lemma 2.5 to get the holonomic sections

$$\Theta_{g_t,E^t}\Big(\widetilde{F}_t(p)\Big) := \Big(p, \widetilde{f}_t(p), D_{E^t}\widetilde{f}_t(p), D^2_{g^t,E^t}\widetilde{f}_t(p)\Big).$$

Since $\widetilde{F}_t \approx F_t$ on a neighborhood U_t of the perturbed core $\psi^t_1(P)$, it follows that, for all $p \in U_t$,

$$\begin{aligned} \mathrm{S}^{\widetilde{f}_t}_{E^t}(p) &= \Big\langle D^2_{g^t,E^t}\widetilde{f}_t(p), N^{\widetilde{f}_t}(p)\Big\rangle \Big(\mathrm{I}^{\widetilde{f}_t}_{E^t}(p)\Big)^{-1} \\ &\approx \Big\langle A_t(p), N^{f_t}(p)\Big\rangle \Big(\mathrm{I}^{f_t}_{E^t}(p)\Big)^{-1} \\ &= \Lambda^t, \end{aligned}$$

where, in the last equality, we use the fact that $\mathrm{I}^{f_t}_{E^t}(p)$ is the identity matrix. So the eigenvalues of $\mathrm{S}^{\widetilde{f}_t}_{E^t}(p)$ are close to λ^t_i for all $p \in U_t$. Thus, as in part (i), letting $r^t_s \colon M \to M$ be the family of retractions into U_t, we obtain our desired homotopy via $\overline{f}_t$, as defined by (2).

(v) Repeat part (iv) with

$$\Lambda^t(p) := \begin{pmatrix} (1-t)k_1^{f_0}(p) + t\,k_1^{f_1}(p) & & 0 \\ & \ddots & \\ 0 & & (1-t)k_n^{f_0}(p) + t\,k_n^{f_1}(p) \end{pmatrix}.$$

□

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