SHORTEST CLOSED CURVE TO CONTAIN A SPHERE IN ITS CONVEX HULL

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Abstract. We show that in Euclidean 3-space any closed curve which contains the unit sphere within its convex hull has length $L \geq 4\pi$, and characterize the case of equality. This result generalizes the authors' recent solution to a conjecture of Zalgaller. Furthermore, for the analogous problem in n dimensions, we include the estimate $L \geq Cn\sqrt{n}$ by Nazarov, which is sharp up to the constant C.

1. Introduction

The *convex hull* of a set X in Euclidean space $\mathbf{R}^3$ is the intersection of all convex sets which contain X. The *inradius* of X is the supremum of the radii of spheres which are contained in X. Here we show:

Theorem 1.1. *Let $\gamma\colon [a,b] \to \mathbf{R}^3$ be a closed rectifiable curve of length L, and r be the inradius of the convex hull of γ. Then*

$$L \geq 4\pi r. \tag{1}$$

Equality holds only if, up to a reparameterization, γ is simple, $\mathcal{C}^{1,1}$, lies on a sphere of radius $\sqrt{2}\,r$, and traces consecutively 4 semicircles of length πr.

In 1996 V. A. Zalgaller [18, 22] conjectured that the above theorem holds subject to the additional assumption that γ lie outside a sphere S of radius r within its convex hull. The length minimizer, called the *baseball curve*, together with S, is shown in Figure 1. Zalgaller's conjecture was proved recently in [15] following earlier work in [13]. Here

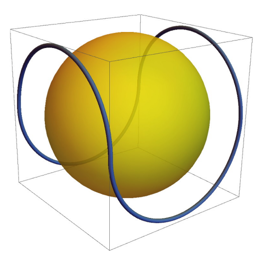

Figure 1. The baseball curve

we refine the methods introduced in those papers to establish the more general result above. Our approach will be similar to that in [15]. We start by setting $r = 1$ and

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assuming that γ has the smallest length among closed curves which contain the unit sphere $\mathbf{S}^2$ within their convex hull [15, Sec 2.]. The *horizon* of γ is the measure in $\mathbf{S}^2$ counted with multiplicity of the set of points $p \in \mathbf{S}^2$ where the affine tangent plane $T_p\mathbf{S}^2$ intersects γ:

$$H(\gamma) := \int_{p\in\mathbf{S}^2} \#\gamma^{-1}(T_p\mathbf{S}^2)\, dp.$$

Since γ is closed, one quickly sees that $\#\gamma^{-1}(T_p\mathbf{S}^2) \geq 2$ for almost every $p \in \mathbf{S}^2$ [13, Lem. 7.1]. Hence $H(\gamma) \geq 8\pi$. The *efficiency* of γ is given by

$$E(\gamma) := \frac{H(\gamma)}{L(\gamma)}.$$

So to establish (1) it suffices to show that $E(\gamma) \leq 2$. To this end we note that for any partition of γ into subcurves γ_i,

$$E(\gamma) = \sum_i \frac{H(\gamma_i)}{L(\gamma)} = \sum_i \frac{L(\gamma_i)}{L(\gamma)} E(\gamma_i).$$

So it suffices to construct a partition with $E(\gamma_i) \leq 2$. Similar to [15], this is achieved by *unfolding* γ into the plane (Section 3), and identifying a collection of subcurves of γ we call *spirals* (Section 4); however, these operations need to be generalized here as they were defined only for curves with $|\gamma| \geq 1$ in [15]. Furthermore, we will show that if $E(\gamma) = 2$, then $|\gamma| \geq 1$. So the rigidity of (1) follows from Zalgaller's conjecture established in [15], and completes the proof of Theorem 1.1 (Section 5).

For curves in $\mathbf{R}^2$ the isoperimetric inequality quickly yields $L \geq 2\pi r$ as the analogue of (1). We will include in the Appendix a version of (1) by F. Nazarov for curves in $\mathbf{R}^n$, which is obtained by covering the unit sphere $\mathbf{S}^{n-1}$ with certain slabs, and applying the correlation inequality [16, 19] to their Gaussian volume. This approach has implications for covering problems for the sphere by congruent disks [5], and yields a new proof of a result of Tikhomirov [20] (Note 5.4). There are many natural optimization problems for convex hull of space curves which remain open, including other questions of Zalgaller [22] which are closely related to well-known problems of Bellman [2–4] in operations research and search theory [1, 12]; see also [13, 15, 17] and references therein.

2. Minimal Inspection Curves

$\mathbf{R}^n$ denotes the n-dimensional Euclidean space with inner product $\langle\cdot,\cdot\rangle$, norm $|\cdot| := \langle\cdot,\cdot\rangle^{1/2}$, and origin o. A *curve* is a continuous rectifiable mapping $\gamma\colon [a,b] \to \mathbf{R}^n$ with length $L = L(\gamma)$. We also use γ to refer to its image $\gamma([a,b])$. If $\gamma(a) = \gamma(b)$ then we say that γ is *closed* and identify $[a,b]$ with the topological circle $\mathbf{R}/(b-a)$. Rectifiable curves may be parameterized with constant speed [6], which we assume is the case throughout this work. In particular all curves below are Lipschitz continuous, and thus differentiable almost everywhere, with $|\gamma'| = L/(b-a)$; see [15, Sec. 2] and references therein for basic facts on rectifiable curves. We say γ is a *(generalized) inspection curve* provided that γ is closed and its convex hull, $\mathrm{conv}(\gamma)$, contains the unit sphere $\mathbf{S}^2$. It follows from Arzela-Ascoli theorem that there exists an inspection curve γ whose length achieves the minimum value among all inspection curves [15, Sec. 2]. Then γ will be called a *minimal* inspection curve. We let *int*, *cl*, and ∂, stand respectively for interior, closure, and boundary.

Lemma 2.1. *Let $\gamma\colon \mathbf{R}/L \to \mathbf{R}^3$ be a minimal inspection curve. Suppose that $\gamma(t) \in \mathrm{int}(\mathrm{conv}(\gamma))$, for some $t \in \mathbf{R}/L$. Then there exists a connected open set $U \subset \mathbf{R}/L$, with $t \in U$, such that γ maps $\mathrm{cl}(U)$ injectively to a line segment with end points on $\partial\,\mathrm{conv}(\gamma)$. In particular, $\gamma(t) = o$ for at most finitely many $t \in \mathbf{R}/L$.*

Proof. Let U be the component of $\gamma^{-1}(\mathrm{int}(\mathrm{conv}(\gamma)))$ which contains t. If $\gamma|_{\mathrm{cl}(U)}$ does not trace a line segment, we may shorten γ by replacing $\gamma(\mathrm{cl}(U))$ with the line segment connecting the end points of $\gamma(\mathrm{cl}(U))$. But this operation preserves $\mathrm{conv}(\gamma)$, as it preserves the points of γ on $\partial\,\mathrm{conv}(\gamma)$. Hence we obtain an inspection curve shorter than γ, which is impossible. If $\gamma(t) = o$, then $L(\gamma|_U) \geq 2$, since $\gamma(U)$ contains a diameter of $\mathbf{S}^2$. So there can be only finitely many such points, since γ is rectifiable. $\square$

We say that t is a *regular* point of a curve γ provided that γ is differentiable at t and $\gamma'(t) \neq 0$. Then the *tangent line* of γ at t is well defined. Since we assume that curves are parameterized with constant speed, they are regular almost everywhere. Furthermore, by Lemma 2.1, all points $t \in \mathbf{R}/L$ with $\gamma(t) \in \mathrm{int}(\mathrm{conv}(\gamma))$ of a minimal inspection curve γ are regular.

Lemma 2.2. *Let $\gamma\colon \mathbf{R}/L \to \mathbf{R}^3$ be a minimal inspection curve, $t \in \mathbf{R}/L$ be a regular point of γ, and ℓ be the tangent line of γ at t. Suppose that ℓ intersects $\mathrm{int}(\mathrm{conv}(\gamma))$. Then there exists an open interval $U \subset \mathbf{R}/L$, with $t \in U$, which is mapped injectively by γ into $\ell \cap \mathrm{int}(\mathrm{conv}(\gamma))$.*

Proof. If $\gamma(t) \in \partial\,\mathrm{conv}(\gamma)$, then either $\gamma'(t)$ or $-\gamma'(t)$ points outside $\mathrm{conv}(\gamma)$. Hence, for some s close to t, $\gamma(s)$ lies outside $\mathrm{conv}(\gamma)$, which is impossible. So $\gamma(t) \in \mathrm{int}(\mathrm{conv}(\gamma))$, in which case Lemma 2.1 completes the proof. $\square$

Combining the last two observations we obtain:

Proposition 2.3. *Let $\gamma\colon \mathbf{R}/L \to \mathbf{R}^3$ be a minimal inspection curve. Then there exists an open set $U \subset \mathbf{R}/L$ such that tangent lines of γ on U do not pass through o. Furthermore if $U \neq \mathbf{R}/L$, then $\mathbf{R}/L \setminus U$ is the disjoint union of a finite number of closed intervals each mapped by γ into a line segment which passes through o and ends on $\partial\,\mathrm{conv}(\gamma)$.*

Proof. Let X be the union of all closed intervals $I \subset \mathbf{R}/L$ such that $\gamma(I)$ is a line segment which passes through o and ends on $\partial\,\mathrm{conv}(\gamma)$. By Lemma 2.1, there are at most finitely many such intervals. Thus X is closed. Let $U := \mathbf{R}/L \setminus X$. By Lemma 2.2, no tangent line of γ at a regular point of U may pass through o, which completes the proof. $\square$

3. Unfolding

Let $\gamma\colon \mathbf{R}/L \to \mathbf{R}^3$ be a minimal inspection curve. We will always assume that 0 is a local minimum point of $|\gamma|$. By Lemma 2.1, γ passes through o at most finitely many times which, if they exist, will be denoted by $0 =: t_0, \dots, t_m := L$. Then the projection $\overline{\gamma}\colon \mathbf{R}/L \to \mathbf{S}^2$, given by $\overline{\gamma} := \gamma/|\gamma|$ is well defined on $\mathbf{R}/L \setminus \{t_k\}$. Furthermore since, by Proposition 2.3, γ traces line segments near t_k, $\overline{\gamma}$ is Lipschitz on each interval (t_{k-1}, t_k). Thus $\overline{\gamma}$ is differentiable almost everywhere on $\mathbf{R}/L$. Consequently, the arclength function

$$\theta(t) := \int_0^t |\overline{\gamma}'(s)| ds$$

is well defined on $[0, L]$ (θ measures the "cone angle" [7] or "vision angle" [8] of γ from the point of view of o). The *unfolding* of γ is the planar curve $\widetilde{\gamma}\colon [0, L] \to \mathbf{R}^2$ defined as

$$\widetilde{\gamma}(t) := |\gamma(t)| e^{i\big(\theta(t)+(k-1)\pi\big)}, \quad \text{for} \quad t \in [t_{k-1}, t_k].$$

Note that $|\gamma| = |\widetilde{\gamma}|$, and whenever γ passes through o, then $\widetilde{\gamma}$ will pass through o as well on a line segment. As in [15], we may also compute that

$$(2) \qquad |\widetilde{\gamma}'| = \big||\gamma|' + i|\gamma|\theta'\big|, \qquad \text{and} \qquad \theta' = |\overline{\gamma}'| = \frac{1}{|\gamma|^2}\sqrt{|\gamma|^2|\gamma'|^2 - \langle \gamma, \gamma'\rangle^2},$$

almost everywhere. It follows that, for almost all $t \in [0, L]$, $|\widetilde{\gamma}'| = |\gamma'| = 1$. So $\widetilde{\gamma}$ is parameterized by arclength, and $L(\gamma) = L(\widetilde{\gamma})$. Hence, by [15, Cor. 3.2], $E(\gamma) = E(\widetilde{\gamma})$ since points of γ with $|\gamma| \leq 1$ make no contribution to $E(\gamma)$. Furthermore, the angles $\alpha := \angle(\gamma, \gamma')$ and $\widetilde{\alpha} := \angle(\widetilde{\gamma}, \widetilde{\gamma}')$ are defined almost everywhere, and

$$(3) \qquad \alpha = \cos^{-1}\big(|\gamma|'\big) = \cos^{-1}\big(|\widetilde{\gamma}|'\big) = \widetilde{\alpha}.$$

Lemma 3.1. *Let $\gamma\colon \mathbf{R}/L \to \mathbf{R}^3$ be a minimal inspection curve. Then $\widetilde{\gamma}$ is locally one-to-one.*

Proof. Let U be as in Proposition 2.3. Then γ and γ' are linearly independent at all regular points of U. So (2) shows that $\theta' > 0$ almost everywhere on U, via Cauchy-Schwarz inequality. Hence θ is strictly increasing on U, which yields that $\widetilde{\gamma}$ is star-shaped with respect to o in a neighborhood of each point of U. Since, by Proposition 2.3, γ traces a line segment on each component of $\mathbf{R}/L \setminus U$, $|\gamma|$ is strictly monotone on each of these components. Hence $\widetilde{\gamma}$ is one-to-one on each component of $\mathbf{R}/L \setminus U$, since $|\widetilde{\gamma}| = |\gamma|$. Finally, $\widetilde{\gamma}$ is one-to-one in a neighborhood of each point of ∂U, since $\widetilde{\gamma}$ is locally star-shaped on U and it maps each component of $\mathbf{R}/L \setminus U$ to a line passing through o. □

A planar curve $\gamma\colon [a, b] \to \mathbf{R}^2$ is *locally convex* provided that it is locally one-to-one and each point $t \in [a, b]$ has a neighborhood $U \subset [a, b]$ such that $\gamma(U)$ lies on the boundary of a convex set. A *side* of a line $\ell \subset \mathbf{R}^2$ is one of the two closed half spaces determined by ℓ. A *local supporting line* ℓ for γ at t is a line passing through $\gamma(t)$ with respect to which $\gamma(U)$ lies on one side. If $\gamma(U)$ lies on a side of ℓ which contains o, then we say that ℓ lies *above* γ. Finally, if γ is locally convex and through each point of it there passes a local support line which lies above γ, then we say that γ is locally convex *with respect to o*. Note that if γ is locally convex with respect to o and passes through o, then γ must trace a line segment near o.

Lemma 3.2. *Let $\gamma\colon \mathbf{R}/L \to \mathbf{R}^3$ be a minimal inspection curve. Then $\widetilde{\gamma}$ is locally convex with respect to o.*

Proof. Let U be as in Proposition 2.3, and $t \in U$. By Lemma 3.1, there exists a neighborhood V of t in U on which $\widetilde{\gamma}$ is one-to-one. Furthermore, $\widetilde{\gamma}(V)$ is star-shaped with respect to o. So connecting the end points of $\widetilde{\gamma}(V)$ to o by line segments yields a simple closed curve. It is shown in the proof of [15, Prop. 4.3] that this curve bounds a convex set, due to minimality of γ. Thus $\widetilde{\gamma}$ is locally convex with respect to o on U. Next suppose that $t \in \partial U$, and let V be a small neighborhood of t in $\mathrm{cl}(U)$. By Proposition 2.3, $\widetilde{\gamma}$ connects one end point of $\widetilde{\gamma}(V)$ to o by tracing a line segment. Connect the other end point of $\widetilde{\gamma}(V)$ to o by another line segment. Then the resulting simple closed curve again bounds a convex set by the argument in the proof of [15, Prop. 4.3]. So $\widetilde{\gamma}$ is locally convex with respect to o on $\mathrm{cl}(U)$. Finally, $\widetilde{\gamma}$ is locally convex with respect to o on the complement of $\mathrm{cl}(U)$, since these regions are mapped to line segments, by Proposition 2.3. □

4. Spiral Decomposition

If $\gamma\colon [a,b] \to \mathbf{R}^2$ is a locally convex curve, parameterized with constant speed, then its one sided derivatives, $\gamma'_{\pm}$, are well-defined everywhere and are nonvanishing [14, Lem. 5.1]. Set $\gamma'(a) := \gamma'_{+}(a)$. We say that $\gamma : [a,b] \to \mathbf{R}^2$ is a *(generalized) spiral* provided that (i) γ is locally convex with respect to o, (ii) $|\gamma|$ is nondecreasing, and (iii) $\langle \gamma(a), \gamma'(a)\rangle = 0$. A spiral is called *strict* if $|\gamma|$ is increasing. A *spiral decomposition* of a curve $\gamma\colon [a,b] \to \mathbf{R}^2$ is a collection U_i of mutually disjoint open subsets of $[a,b]$ such that (i) $\gamma|_{\mathrm{cl}(U_i)}$ is a strict spiral, after switching the direction of $\gamma|_{\mathrm{cl}(U_i)}$ if necessary, and (ii) $|\gamma|' = 0$ almost everywhere on $[a,b] \setminus \cup_i \mathrm{cl}(U_i)$.

Lemma 4.1. *Let $\gamma\colon \mathbf{R}/L \to \mathbf{R}^3$ be a minimal inspection curve. Then $\widetilde{\gamma}$ admits a spiral decomposition.*

Proof. The argument follows the same outline as in [15, Prop. 5.2], with minor modifications. Recall that we assume 0 is a local minimum point of $|\gamma|$. If $|\gamma(0)| > 0$, then it follows that $\widetilde{\alpha}(0) = \widetilde{\alpha}(L) = \pi/2$. Otherwise, $|\widetilde{\gamma}(0)| = |\widetilde{\gamma}(L)| = 0$, since $|\gamma| = |\widetilde{\gamma}|$. Let X be the set of points $t \in [0,L]$ such that $\widetilde{\gamma}$ has a local support line at $\widetilde{\gamma}(t)$ which is orthogonal to $\widetilde{\gamma}(t)$, or $|\widetilde{\gamma}(t)| = 0$. Then 0, $L \in X$ and $|\widetilde{\gamma}|' = 0$ almost everywhere on X. Also note that X is closed, since the limit of any sequence of support lines of a convex body is a support line, and the set of points with $|\widetilde{\gamma}(t)| = 0$ is compact. Consequently each component U of $[0,L] \setminus X$ is an open subinterval of $[0,L]$. It remains to show that $\widetilde{\gamma}|_{\mathrm{cl}(U)}$ is a spiral. By Lemma 3.2, $\widetilde{\gamma}|_{\mathrm{cl}(U)}$ is locally convex with respect to o. Furthermore, as argued in the proof of [15, Prop. 5.2], $|\widetilde{\gamma}|'$ is always positive or always negative at differentiable points of $|\widetilde{\gamma}|$ on U. So we may suppose that $|\widetilde{\gamma}|$ is increasing on U, after switching the direction of $\widetilde{\gamma}|_{\mathrm{cl}(U)}$ if necessary. Finally, let $x \in \partial U$ be the initial point of $\widetilde{\gamma}|_{\mathrm{cl}(U)}$. If $|\widetilde{\gamma}(x)| = 0$, then $\widetilde{\gamma}|_{\mathrm{cl}(U)}$ is a spiral. If $|\widetilde{\gamma}(x)| > 0$, it follows that $\widetilde{\gamma}(x)$ it orthogonal to $\widetilde{\gamma}'_{+}(x)$, which again shows that $\widetilde{\gamma}|_{\mathrm{cl}(U)}$ is a spiral and completes the proof. □

Let $\mathbf{S}^1$ denote the unit circle in $\mathbf{R}^2$. The last observation quickly yields:

Lemma 4.2. *Let γ, $\widetilde{\gamma}$ be as in Lemma 4.1 and $\sigma\colon [a,b] \to \mathbf{R}^2$ be a spiral in the decomposition of $\widetilde{\gamma}$. Let $t \in [a,b]$ be a regular point of both σ and γ, and ℓ be the tangent line of σ at t. Suppose that ℓ crosses $\mathbf{S}^1$. Then $\sigma([a,t])$ lies on ℓ.*

Proof. Let $\overline{\ell}$ be the tangent line of γ at t. If ℓ crosses $\mathbf{S}^1$, then $\overline{\ell}$ crosses $\mathbf{S}^2$, by (3). In particular, $\overline{\ell}$ intersects the interior of $\mathrm{conv}(\gamma)$. Then Lemma 2.2 completes the proof. □

The key point in the proof of Theorem 1.1 is:

Proposition 4.3. *Let $\sigma\colon [a,b] \to \mathbf{R}^2$ be a spiral in the unfolding of a minimal inspection curve. Then $E(\sigma) \leq 2$. Furthermore, if $|\sigma(a)| < 1$, then $E(\sigma) < 2$.*

Proof. If $|\sigma(a)| \geq 1$, then $E(\sigma) \leq 2$ by [15, Prop. 2.7]. So we assume $|\sigma(a)| < 1$. We may also assume that $|\sigma(b)| > 1$ for otherwise $H(\sigma) = 0$ which yields $E(\sigma) = 0$. Let b' be the supremum of $t \in [a,b]$ such that $\sigma([a,t])$ is a line segment. By Lemma 2.1, $|\sigma(b')| \geq 1$. We may assume that $\sigma(a)$ lies on the nonnegative portion of the y-axis, and $\sigma([a,b'])$ lies to the right of the y-axis, see Figure 2. If $b' < b$, then we may choose

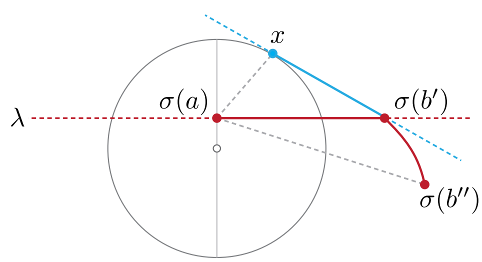

FIGURE 2. Construction of the competing curve

$b' < b'' < b$ such that $\sigma([b',b''])$ is convex, and lies to the right of the y-axis. Since σ is locally convex with respect to o, $\sigma([b',b''])$ lies below the line λ spanned by $\sigma([a,b'])$, if $|\sigma(a)| > 0$. If $|\sigma(a)| = 0$, we may still assume that $\sigma([b',b''])$ lies below λ after a reflection. Consider the line which passes through $\sigma(b')$ and is tangent to the upper half of $\mathbf{S}^1$, say at a point x. Let τ be the curve obtained by joining the line segment $x\sigma(b')$ to the beginning of $\sigma|_{[b',b]}$. We will show that (i) τ is a spiral, and (ii) $E(\sigma) < E(\tau)$. Then we are done, because $E(\tau) \leq 2$ since its initial height is ≥ 1.

First we check that τ is a spiral. This is obvious if $b' = b$. So assume that $b' < b$, and let $b' < b'' < b$ be as defined above. It suffices to check that τ is locally convex at $\sigma(b')$. Connect the end points of the portion $x\sigma(b'')$ of τ to $\sigma(a)$ to obtain a closed curve Γ. Note that Γ is simple since $x\sigma(b')$ lies above λ while $\sigma([b',b''])$ lies below it. Let θ be the interior angle of Γ at $\sigma(b')$. We need to show that $\theta \leq \pi$. To this end let $t_i \in (b',b'')$ be a sequence of regular points of σ converging to b', and ℓ_i be tangent lines of σ at t_i. Then ℓ_i converge to a support line of $\sigma([b',b''])$ at $\sigma(b')$, which we call ℓ. By Lemma 4.2, ℓ_i do not cross $\mathbf{S}^1$. Consequently ℓ does not cross $\mathbf{S}^1$ either. So ℓ also supports $x\sigma(b')$. Hence ℓ is a support line of Γ at $\sigma(b')$, which yields that $\theta \leq \pi$ as desired.

It remains to check that $E(\sigma) < E(\tau)$. To see this consider the triangle $\sigma(a)x\sigma(b')$. The interior angle of this triangle at x is $\geq \pi/2$, since $\sigma(a)$ lies on the nonnegative portion of the y-axis. Hence $|x\sigma(b')| < |\sigma(a)\sigma(b')|$, which yields $L(\tau) < L(\sigma)$. On the other hand, tangent planes of $\mathbf{S}^2$ intersect $\mathbf{R}^2 \simeq \mathbf{R}^2 \times \{0\} \subset \mathbf{R}^3$ in lines which do not cross $\mathbf{S}^1$, and any such line has exactly the same number of transverse intersections with σ as it does with τ. Hence $H(\tau) = H(\sigma)$ by definition of horizon. So $E(\sigma) < E(\tau)$ as desired. □

5. Proof of Theorem 1.1

Set $r = 1$ and let $\gamma\colon \mathbf{R}/L \to \mathbf{R}^3$ be a minimal inspection curve, as discussed in Section 2. To establish (1) it suffices to show then that $E(\gamma) \leq 2$, as outlined in Section 1. In Section 3 we established that $E(\gamma) = E(\widetilde{\gamma})$ where $\widetilde{\gamma}\colon [0, L] \to \mathbf{R}^2$ is the unfolding of γ. By Lemma 4.1, $\widetilde{\gamma}$ admits a spiral decomposition, generated by a collection of mutually disjoint open sets $U_i \subset [0, L]$, $i \in I$. Set $U_0 := [0, L] \setminus \cup_i \mathrm{cl}(U_i)$, and let $\widetilde{\gamma}_i := \widetilde{\gamma}|_{\mathrm{cl}(U_i)}$, $\widetilde{\gamma}_0 := \widetilde{\gamma}|_{U_0}$. As in the proof of Zalgaller's conjecture in [15, Sec. 10], we have

$$E(\widetilde{\gamma}) = \frac{H(\widetilde{\gamma})}{L(\widetilde{\gamma})} = \frac{1}{L(\widetilde{\gamma})}\sum_i H(\widetilde{\gamma}_i) = \frac{1}{L(\widetilde{\gamma})}\left(L(\widetilde{\gamma}_0)E(\widetilde{\gamma}_0) + \sum_i L(\widetilde{\gamma}_i)E(\widetilde{\gamma}_i)\right). \tag{4}$$

By Lemma 2.2, every point $t \in [0, L]$ with $|\gamma(t)| < 1$ lies on a line segment in γ with end points on $\mathbf{S}^2$, and thus $\widetilde{\gamma}(t)$ belongs to a strict spiral (with origin of the spiral corresponding to the midpoint of that line segment). So $|\widetilde{\gamma}_0| \geq 1$. Then, as described in [15, Sec. 10], $E(\widetilde{\gamma}_0) \leq 2$. Furthermore $E(\widetilde{\gamma}_i) \leq 2$ for all i by Proposition 4.3. So $E(\widetilde{\gamma}) \leq 2$ by (4), as desired. To characterize the case of equality in (1), note that by (4), if $E(\widetilde{\gamma}) = 2$ then $E(\widetilde{\gamma}_i) = 2$. Consequently, by Proposition 4.3, $|\widetilde{\gamma}_i| \geq 1$. So $|\widetilde{\gamma}| \geq 1$, which yields $|\gamma| \geq 1$. Hence, by the proof of Zalgaller's conjecture [15, Thm. 1.1], γ is the baseball curve.

Appendix: Higher Dimensions

Here we establish a higher dimensional version of (1) due to Fedor Nazarov:

Theorem 5.1 (Nazarov)**.** *Let $\gamma\colon [a, b] \to \mathbf{R}^n$ be a curve of length L, and r be the inradius of the convex hull of γ. Then*

$$L \geq Cn\sqrt{n}\, r, \tag{5}$$

where $C > 0$ is an absolute constant.

By *absolute constant* here we mean that C does not depend on n or γ. A Hamiltonian path in the edge graph of the *cross polytope*, i.e., the unit ball with respect to the L^1-norm in $\mathbf{R}^n$, gives an example of a curve with $L \leq 2n\sqrt{2n}r$ [2]. Thus (5) is sharp up to the constant C. To establish (5), we may set $r = 1$. Furthermore, we may assume that n is even. Indeed suppose that (5) holds for even n. If n is odd and bigger than 1, then we may project γ into $\mathbf{R}^{n-1}$ to obtain $L \geq C(n-1)^{3/2} \geq (C/2)n^{3/2}$. Finally, it is enough to show that if $L \leq Cn\sqrt{n}$, for some absolute constant C, then the inradius of $\mathrm{conv}(\gamma) \leq 1$, which means that there exists $u \in \mathbf{S}^{n-1}$ such that $\langle\gamma(t), u\rangle \leq 1$ for all $t \in [a, b]$. Equivalently, if $L \leq 2n\sqrt{n}$, then $\langle\gamma(t), u\rangle \leq C/2$. In summary, it suffices to show:

Proposition 5.2. *Let* $\gamma\colon [a,b]\to \mathbf{R}^{2n}$ *be a curve of length* $\leq 2n\sqrt{n}$*. Then there exists* $u\in \mathbf{S}^{2n-1}$ *such that* $\langle \gamma(t), u\rangle \leq C$ *for all* $t\in[a,b]$*.*

To prove the above proposition, we again assume that γ has constant speed. Let $t_i \in [a,b]$, $i=0,\dots,n$, be equidistant points with $t_0:=a$, $t_n:=b$, and set $s_i:=(t_{i-1}+t_i)/2$ for $i=1,\dots,n$. Let H be an n-dimensional subspace of $\mathbf{R}^{2n}$ which is orthogonal to each $\gamma(s_i)$, and $\overline{\gamma}$ be the projection of γ into H. Then $\overline{\gamma}|_{[t_{i-1},s_i]}$, $\overline{\gamma}|_{[s_i,t_i]}$ are curves of length $\leq \sqrt{n}$ with one end at o, since γ has constant speed. So, identifying H with $\mathbf{R}^n$, we have reduced Proposition 5.2 to:

Proposition 5.3. *Let* $\gamma_i\colon [a,b]\to\mathbf{R}^n$*,* $i=1,\dots,2n$*, be curves of length* $\leq\sqrt{n}$ *with* $\gamma_i(a)=o$*. Then there exists* $u\in\mathbf{S}^{n-1}$ *such that* $\langle\gamma_i(t),u\rangle\leq C$ *for all* $t\in[a,b]$*.*

To prove the last proposition we employ the standard Gaussian measure, which is defined for Borel sets $A\subset\mathbf{R}^n$ as

$$\mu(A):=\frac{1}{(\sqrt{2\pi})^n}\int_A e^{-|x|^2/2}\,d\lambda(x),$$

where λ is the n-dimensional Lebesgue measure. We also record that if K_i are a family of convex sets which are symmetric with respect to o, then

$$\mu\left(\bigcap_i K_i\right)\geq\prod_i\mu(K_i) \tag{6}$$

by the Gaussian correlation inequality [16, 19]. Here we need this fact only for slabs, which had been established in [21].

Proof of Proposition 5.3. We set $[a,b]=[0,1]$ and assume that γ_i have constant speed. For every $t\in[0,1]$ and i there exist vectors $v_{ik}(t)\in\mathbf{R}^n$, such that

$$\gamma_i(t):=\sum_{k=1}^{\infty}v_{ik}(t),\qquad\text{and}\qquad |v_{ik}(t)|\leq\frac{\sqrt{n}}{2^k}.$$

To generate these vectors, set $t_0:=0$, and let $t_k:=t_{k-1}-1/2^k$, if $t<t_{k-1}$, and $t_k:=t_{k-1}+1/2^k$ otherwise. Then we set $v_{ik}(t):=\gamma_i(t_k)-\gamma_i(t_{k-1})$. Note that each $v_{ik}(t)$ is chosen from a set V_{ik}, of cardinality 2^{k-1}, which is independent of t. Now consider the slabs

$$S(v):=\left\{x\in\mathbf{R}^n \;\middle|\; \big|\langle x,v\rangle\big|\leq\frac{\sqrt{n}}{k^2}\right\},\quad v\in V_{ik},$$

which have width $2(\sqrt{n}/k^2)/|v|\geq 2(2^k/k^2)$, and set

$$A:=\bigcap_{i=1}^{2n}\bigcap_{k=1}^{\infty}\bigcap_{v\in V_{ik}}S(v).$$

By Fubini's theorem, and a standard estimate for the Gaussian integral,

$$\mu\big(S(v)\big)\geq\frac{1}{\sqrt{2\pi}}\int_{-a_k}^{a_k}e^{-t^2/2}\,dt\geq 1-e^{-a_k^2/2},$$

where $a_k := 2^k/k^2$. So by (6),

$$\mu(A) \geq \prod_{i=1}^{2n} \prod_{k=1}^{\infty} \prod_{v \in V_{ik}} \mu\big(S(v)\big) \geq \left(\prod_{k=1}^{\infty} \left(1 - e^{-a_k^2/2}\right)^{2^{k-1}} \right)^{2n}.$$

Since $\ln(1-e^{-x}) \geq -2e^{-x}$ for $x \geq 32/81$, which is the smallest value of $a_k^2/2$ (achieved for $k=3$), we have

$$\begin{aligned} \prod_{k=1}^{\infty} \left(1 - e^{-a_k^2/2}\right)^{2^{k-1}} &= \exp\left(\sum_{k=1}^{\infty} 2^{k-1} \ln\left(1 - e^{-a_k^2/2}\right) \right) \\ &\geq \exp\left(-\sum_{k=1}^{\infty} 2^k e^{-a_k^2/2} \right) =: \sqrt{\delta} > 0. \end{aligned}$$

So we conclude that $\mu(A) \geq \delta^n$ where $\delta > 0$ is an absolute constant. Next note that, if B_r^n is the ball of radius r centered at o in $\mathbf{R}^n$, with volume $|B_r^n|$, then

$$\mu(B_r^n) \leq \frac{|B_r^n|}{(\sqrt{2\pi})^n} = \left(\frac{\sqrt{e}\, r}{\sqrt{n}}\right)^n \frac{|B_{\sqrt{n}}^n|}{(\sqrt{2\pi})^n(\sqrt{e})^n} \leq \left(\frac{\sqrt{e}\, r}{\sqrt{n}}\right)^n \mu(B_{\sqrt{n}}^n) \leq \left(\frac{\sqrt{e}\, r}{\sqrt{n}}\right)^n.$$

So if $r := \delta\sqrt{n}/\sqrt{e}$, then $\mu(B_r^n) \leq \delta^n \leq \mu(A)$. Consequently, $A \not\subset \operatorname{int}(B_r^n)$ which means that there exists $u_0 \in A$ with $|u_0| \geq r$. Now setting $u := u_0/|u_0|$, we have

$$\langle \gamma_i(t), u \rangle = \sum_{k=1}^{\infty} \langle v_{ik}, u \rangle \leq \frac{1}{r} \sum_{k=1}^{\infty} \langle v_{ik}, u_0 \rangle \leq \frac{\sqrt{e}}{\delta\sqrt{n}} \sum_{k=1}^{\infty} \frac{\sqrt{n}}{k^2} \leq \frac{2\sqrt{e}}{\delta} =: C,$$

as desired. □

Note 5.4. When γ_i in Proposition 5.3 trace lines segments, we obtain the following result in discrete geometry: if $N \leq 2n$ points in $\mathbf{R}^n$ contain $\mathbf{S}^{n-1}$ within their convex hull, then at least one of them has distance $\geq \sqrt{n}/C$ from o. Equivalently, if $N \leq 2n$ disks of geodesic radius ρ cover $\mathbf{S}^{n-1}$, then $\cos(\rho) \leq C/\sqrt{n}$, which had been observed earlier by Tikhomirov [20]. Furthermore, proof of Proposition 5.3 allows an estimate for C as follows. If γ_i trace line segments, we may set $k=1$. Then $\mu(S(v)) \geq \int_{-2}^{2} e^{-t^2/2} dt/\sqrt{2\pi} \geq 0.95$. So $\delta = (0.95)^2$, which yields $C = \delta/(2\sqrt{e}) \simeq 3.65$. It has been conjectured that the optimal value of C is 1, which would correspond to the case where the points form the vertices of a cross polytope [5, Conj.1.3]. This has been shown only for $n=3$ [10], see [11, p. 34], and $n=4$ [9].

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