Deformations of Unbounded Convex Bodies and Hypersurfaces

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Let $K \subset \mathbf{R}^n$ be a *convex body*, i.e., a closed convex set with interior points.

If K is compact, then the Minkowski sum yields a canonical homotopy to the unit ball:

$$K_t := (1-t)K + t\,\mathbf{B}^n$$

Thus we obtain a deformation retraction of the space of compact convex bodies onto $\mathbf{B^n}$, with respect to the Hausdorff metric.

Also note that ∂K_t yields a deformation retraction of the space of compact convex hypersurfaces onto $\partial\mathbf{B^n} = \mathbf{S}^{n-1}$."

So the space of *compact* convex bodies (and hypersurfaces) is contractible (with respect to the Hausdorff topology).

Question (H. Rosenberg)

How about the space of *unbounded* convex bodies (and hypersurfaces)? What is the topological structure of this space?

Some Background:

Theorem (Gluck and Pan)

If a pair of positively curved compact connected surfaces with boundary in $\mathbf{R}^3$ *are regularly homotopic (resp. isotopic), then they are regularly homotopic (resp. isotopic) through surfaces of positive curvature.*

Proof.

Explicit construction of the homotopy. □

Theorem (Ghomi and Kossowski)

Same result holds also in $\mathbf{R}^n$ *for hypersurfaces with prescribed signs of principal curvatures.*

Proof.

Application of the *h*-principle in the sense of Gromov; specifically, the holonomic approximation theorem of Eliashberg and Mishachev. □

Let $\mathcal{K}^n$ be the space of (genuine) unbounded convex bodies in $\mathbf{R}^n$, i.e., unbounded closed convex sets with interior points, whose boundary are homeomorphic to $\mathbf{R}^{n-1}$.

We like to show that each $K \in \mathcal{K}^n$ may be deformed canonically to a half-space whose boundary passes through the origin of $\mathbf{R}^n$.

First we need to choose a suitable topology on $\mathcal{K}^n$.

Recall that the space of *compact* convex bodies has a natural topology given by hausdorff distance:

$$\mathrm{dist}_h(K, L) := \inf\{r \geqslant 0 \mid K \subset L + r\mathbf{B}^n \quad \text{and} \quad L \subset K + r\mathbf{B}^n\}$$

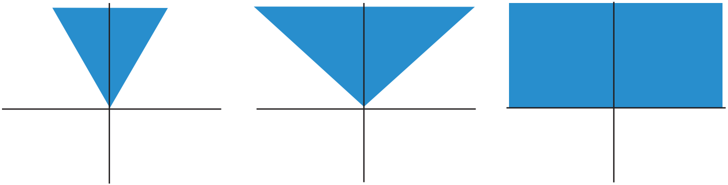
Although, the hausdorff distance is not well-defined when K or L are unbounded, we may nevertheless define a Hausdorff topology on $\mathcal{K}^n$ by using the neighborhoods $K + \epsilon B^n$ as the basis elements.

But this topology will be much too rigid at infinity to allow the deformations we seek ...

Example

Consider the family of convex bodies $K_i \subset \mathbf{R}^2$ given by

$$y \geqslant \frac{|x|}{i}$$



This family does **not** converge to the upper half-plane with respect to the Hausdorff topology, as $i \to \infty$.

One way to relax the Hausdorff topology is as follows:

Let $\pi\colon \mathbf{S}^n - \{0,0,1\} \to \mathbf{R}^n$ be the stereographic projection. Then we define the *bounded Hausdorff distance* as follows

$$\mathrm{dist}_{bh}(K, L) := \mathrm{dist}_h\left(\pi^{-1}(K), \pi^{-1}(L)\right).$$

The induced topology on $\mathbf{R}^n$ is called *bounded Hausdorff topology* and also coincides with Attouch-Wets, Fells, and Wijsman topologies studied in convex analysis.

It is also useful to note that

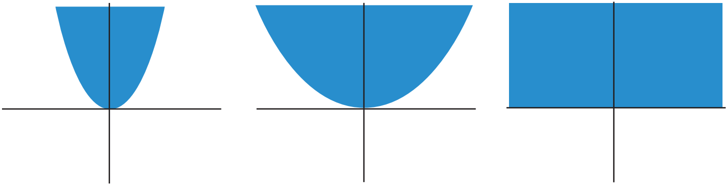
$$K \xrightarrow{bh} L \quad \Longleftrightarrow \quad K \cap r\mathbf{B}^n \xrightarrow{h} L \cap r\mathbf{B}^n \quad \text{for all } r > 0$$

Under this topology $K_i \subset \mathbf{R}^2$ given by $y \geqslant |x|/i$ converges to $y \geqslant 0$; however, the bounded hausdorff topology still has some drawbacks ...

Example

Consider $K_i \subset \mathbf{R}^2$ given by

$$y \geqslant \frac{x^2}{i}$$



This family converges to the half space $y \geqslant 0$ with respect to the bounded Hausdorff topology as $i \longrightarrow \infty$; however, the total curvature of each K_i is 2π (which does not converge to 0).

Thus the bounded Hausdorff topology is too weak or permissive as far as maintaining the continuity of curvature is concerned.

So our ideal topology lies somewhere in between Hausdorff and bounded Hausdorff topologies.

To find this topology we proceed as follows ...

The *recession cone* of a convex body $K \subset \mathbf{R}^n$ is defined as

$$\mathrm{rc}(K) := \{y \in \mathbf{R}^n \mid K + y \subset K\}.$$

It is well-known that $\mathrm{rc}(K)$ is a closed convex cone.

Note that $\mathrm{rc}(K)$ represents the asymptotic behavior of K or what it "looks like" when viewed from far away.

We define the *asymptotic distance* as follows;

$$\mathrm{dist}_a(K, L) := \mathrm{dist}_{bh}(K, L) + \mathrm{dist}_{bh}(\mathrm{rc}(K), \mathrm{rc}(L)).$$

The induced topology will be called the *asymptotic topology*.

The curvature will behave well under this topology ...

The *normal cone* of a convex body K is the collection of all outward normal vectors to support hyperplanes of K:

$$\mathrm{nc}(K) := \{v \in \mathbf{R}^n \mid \langle x-p, v\rangle \leqslant 0, \quad \text{for all } x \in K \text{ and some } p \in \partial K\}.$$

The *total curvature* of K is the measure in $\mathbf{S}^{n-1}$ of the unit elements of $\mathrm{nc}(K)$:

$$\tau(K) := \mu\Big(\mathrm{nc}(K) \cap \mathbf{S}^{n-1}\Big)$$

We claim that if $K_i \xrightarrow{a} K$, then $\tau(K_i) \to \tau(K)$.

If $C \subset \mathbf{R}^n$ is a closed convex cone, then its *dual cone* is given by $C^* = \mathrm{nc}(C)$. Further, it is well-known that $C^{**} = \mathrm{cl}(C)$. It follows from these results, via the notion of *barrier cone*, that

$$\mathrm{rc}(K)^* = \mathrm{cl}\left(nc(K)\right).$$

Another basic fact is that the operation $C \mapsto C^*$ is continuous with respect to the bounded hausdorff topology. So

$$\begin{aligned} K_i \xrightarrow{a} K \quad &\Longrightarrow \quad \mathrm{rc}(K_i) \xrightarrow{bh} \mathrm{rc}(K) \\ &\Longrightarrow \quad \mathrm{rc}(K_i)^* \xrightarrow{bh} \mathrm{rc}(K)^* \\ &\Longrightarrow \quad \mathrm{cl}(\mathrm{nc}(K_i)) \xrightarrow{bh} \mathrm{cl}(\mathrm{nc}(K)) \\ &\Longrightarrow \quad \tau(K_i) \to \tau(K) \end{aligned}$$

Now we can finally state the main result of this talk:

Let $\mathcal{H}^n$ denote the space of closed half-spaces whose boundary passes through the origin of $\mathbf{R}^n$.

Theorem

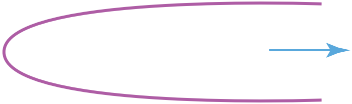
There exists a regularity preserving strong deformation retraction $\mathcal{K}^n \to \mathcal{H}^n$ *with respect to the asymptotic topology.*

In particular, $\mathcal{K}^n/SO(n)$ is contractible.

Furthermore, the subpsaces of $\mathcal{K}^n$ consiting of smooth, strictly convex, or positively curved elements are each contractible as well.

It is not clear, if not impossible, how one could flatten elements of $\mathcal{K}^n$ using PDE's or geometric flows:

For instance, there are hypersurfaces such the famous *grim reaper*, given by $x = \log(\cos(y))$ which move by translations under the mean curvature flow and thus never become flat.



Such singluar solutions also exist in higher dimensions and for Gauss flow.

Now we proceed towards constructing the deformation retraction $\mathcal{K}^n \to \mathcal{H}^n$.

More explicitly we construct a continuous mapping $\mathcal{K}^n \times [0,1] \to \mathcal{K}^n$, $(K,t) \mapsto K_t$ such that $K_0 = K$ and K_1 is a halfspace whose boundary passes through the origin.

And continuity here is with respect to the asymptotic topology.

First we construct a map $\mathrm{cd}\colon \mathcal{K}^n \to \mathbf{S}^{n-1}$ which we call the *central direction*.

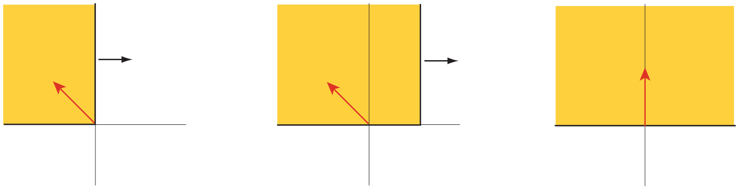
$$\mathrm{cd}(K) := \frac{\int_{\overline{\mathrm{rc}}(K)} x \, d\omega_{m-1}}{\| \int_{\overline{\mathrm{rc}}(K)} x \, d\omega_{m-1} \|},$$

where $\overline{\mathrm{rc}}(K) := \mathrm{rc}(K) \cap \mathbf{S}^{n-1}$ and m is the dimension of the affine hull of $\mathrm{rc}(K)$. Then

- $\mathrm{cd}\colon \mathcal{K}^n \to \mathbf{S}^{n-1}$ is well-defined and continuous with respect to the asymptotic topology.
- $-\mathrm{cd}(K) \in \mathrm{nc}(K)$.
- Let ∂H be the support hyperplane of K orthogonal to $\mathrm{cd}(K)$. Then $\partial H \cap K$ contains a half-line only when it contains the whole line.

Note

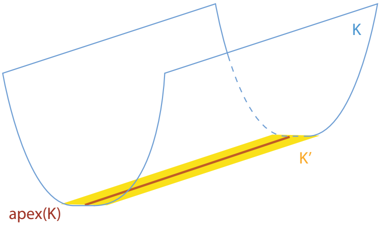
cd: $\mathcal{K}^n \to \mathbf{S}^{n-1}$ is **not** continuous with respect to the bounded Hausdorff topology:



Definition

Let ∂H be the support hyperplane of $K \in \mathcal{K}^n$ with outward normal $-\operatorname{cd}(K)$, and set $K' := \partial H \cap K$. Then $K' = \overline{K}' + L'$, where L' is an affine subspace of $\mathbf{R}^n$ and $\overline{K}'$ is a compact convex set. We define:

$$\operatorname{apex}(K) := \operatorname{cm}(\overline{K}') + L'.$$



Now on to the construction of the defomation retraction $\mathcal{K}^n \to \mathcal{H}^n$:

1. Translations
2. Flatennings

Step 1: Translations

Let $p(K)$ be the closest point of $\mathrm{apex}(K)$ to the origin of $\mathbf{R}^n$, and for each $K \in \mathcal{K}^n$ set

$$K_t := K - t\, p(K).$$

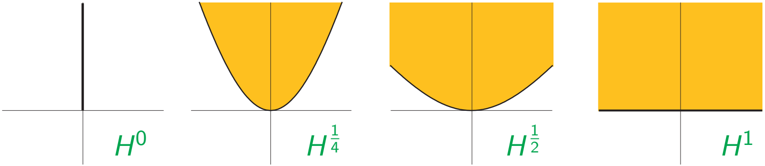
This gives a strong deformation retraction $\mathcal{K}^n \to \overline{\mathcal{K}}^n$, where $\overline{\mathcal{K}}^n$ is the space of those bodies $K \in \mathcal{K}^n$ with $o \in \mathrm{apex}(K)$.

Step 2: Flatennings

Now we construct a deformation retraction $\overline{\mathcal{K}}^n \to \mathcal{H}^n$: Let $H^t \subset \mathbf{R}^n$ be the family of hyperboloidal convex bodies given by

$$x_n \geqslant \left(\sqrt{1 + \sum_{i=1}^{n-1} x_i^2} - 1\right) \frac{1-t}{t}$$

for $0 < t \leqslant 1$, and set H^0 equal to the upper-half of the x_n axis.



Next let H_u^t be the object which is obtained by a rotation of H^t about o so that its central direction coincides with u. Then, for $K \in \overline{\mathcal{K}}^n$, we define

$$K_t := K + H^t_{\mathrm{cd}(K)}.$$

Definition

We say that a unit vector $u \in \mathbf{S}^{n-1}$ is a *balanced support vector* of $K \in \mathcal{K}^n$ provided that

- $u \in \mathrm{nc}(K)$.
- $-u \in \mathrm{rc}(K)$.
- If ∂H is the support hyperplane of K orthogonal to u, then $\partial H \cap K$ contains a half-line only when it contains the whole line.

Example

$-cd(K)$ is a balanced support vector of K.

Let $\mathcal{K}^n_+ \subset \mathcal{K}^n$ be the collection of bodies which are strictly convex at (at least) one point.

Let $\mathcal{K}^n_u$ (resp. $(\mathcal{K}^n_+)_u$) be the collection of bodies in $\mathcal{K}^n$ (resp. $\mathcal{K}^n_+$) with balanced support u.

Proposition (Continuity of Minkowski Sums)

For any pair of convex bodies $K_0, K_1 \in (\mathcal{K}^n_+)_u$, $K_0 + K_1 \in (\mathcal{K}^n_+)_u$. Furthermore, the Minkowski addition

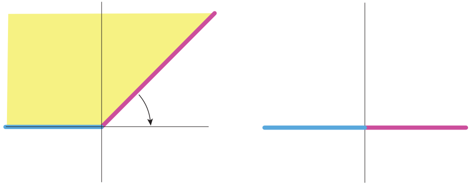
$$+\colon (\mathcal{K}^n_+)_u \times (\mathcal{K}^n_+)_u \to (\mathcal{K}^n_+)_u$$

is asymptotically continuous. Furthermore, for any pair of convex bodies $K_0 \in \mathcal{K}^n_u$, and $K_1 \in (\mathcal{K}^n_+)_u$, $K_0 + K_1 \in \mathcal{K}^n_u$, and for any fixed $K_0 \in \mathcal{K}^n_u$, the mapping

$$K_0 + (\cdot)\colon (\mathcal{K}^n_+)_u \to \mathcal{K}^n_u$$

is asymptotically continuous.

The Minkowski addition is not continuous on $\mathcal{K}^n$, even with respect to the bounded-Hausdorff topology.



Proposition (Regularity of Minkowski Sums)

Let K_0, $K_1 \subset \mathbf{R}^n$ be convex bodies, and suppose that

$$K := K_0 + K_1$$

is closed. Then K is also a convex body, and the following hold:

1. *If K_0, K_1 are strictly convex, then so is K;*
2. *If K_0 is C^1, then so is K;*
3. *If K_0, K_1 are $C^{2\leqslant k\leqslant\infty}$ and K_0 is positively curved, then K is also C^k;*
4. *If K_0, K_1 are C^ω and K_0 is positively curved, then K is also C^ω;*
5. *If K_0, K_1 have positive curvature, then so does K.*

Thank You!