Durer's Problem on Unfoldability of Convex Polyhedra

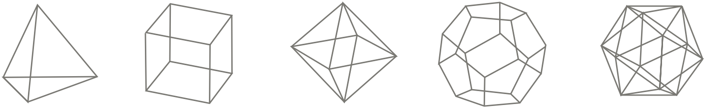
Mohammad Ghomi

Georgia Tech

January 23, 2019

Neolithic carved stones, Scotland, 2000 BC

Examples of (edge) unfoldings of Platonic solids



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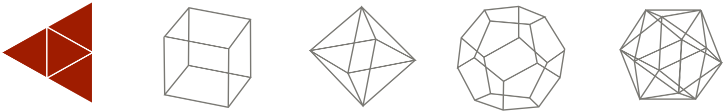

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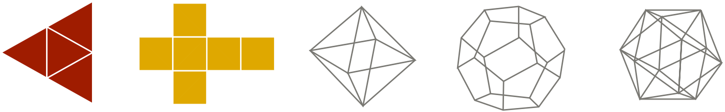

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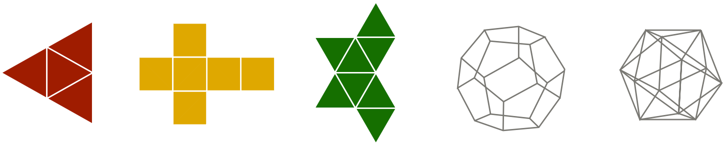

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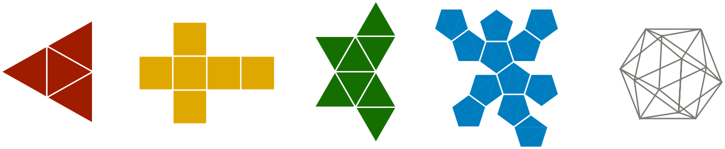

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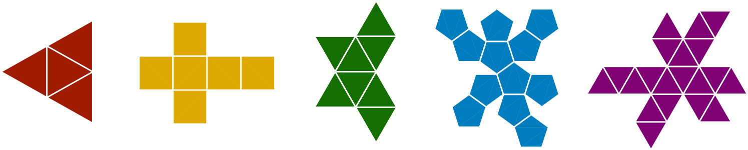

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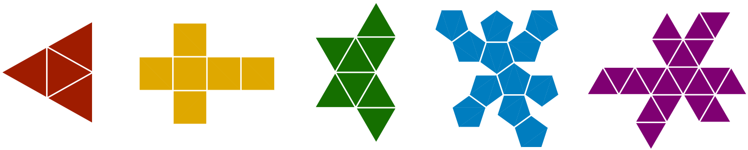
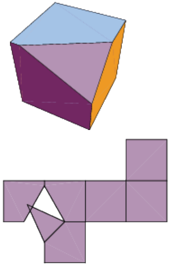
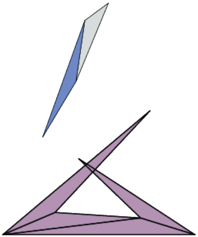


Every unfolding of a platonic solid is simple (non-overlapping).

Nonsimple unfoldings

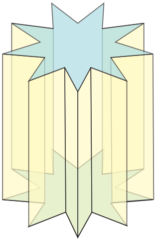
But not every unfolding of every convex polyhedron is simple:

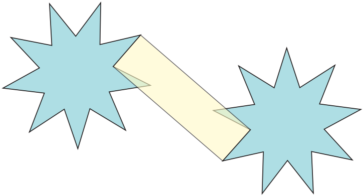




Nonsimple unfoldings

There are even some *nonconvex* polyhedra which do not have any simple unfoldings:





But no such example has been found for convex polyhedra.

Dürer's Problem

Problem

Does every convex polyhedron have a simple unfolding?

Outline of the Talk

1. Some history (geometers and painters)
2. Survey of known results (almost all negative)
3. A new result (positive)
4. A newer result (negative)
5. How to construct a counterexample to Durer's conjecture (if one exists)

Albrecht Dürer (1471-1528)

“This I drew, using a mirror;
it is my own likeness,
in the year 1484,
when I was still a child.”

Albrecht Dürer (1471-1528)

Albrecht Dürer (1471-1528)

Albrecht Dürer (1471-1528)

“Thus I, Albrecht Duerer from Nuremburg,
painted myself with indelible colours
at the age of 28 years.”

Albrecht Dürer (1471-1528)

Albrecht Dürer (1471-1528)

Trips to Venice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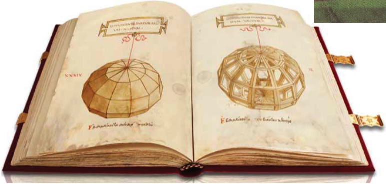

``The secret of perspective"

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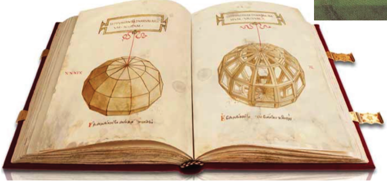

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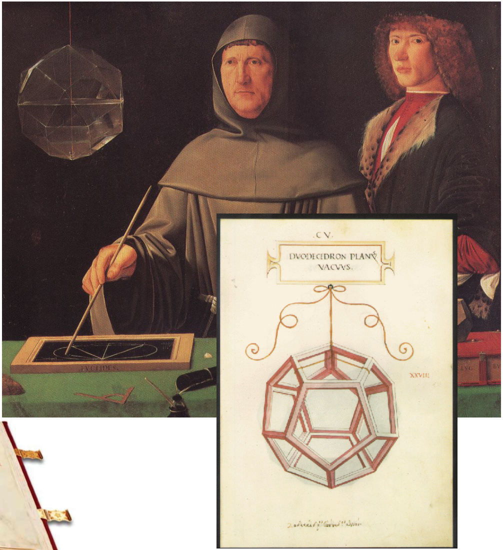

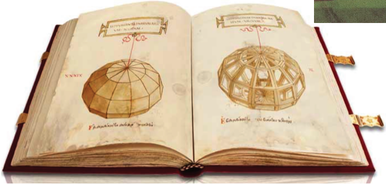

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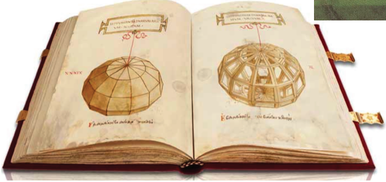

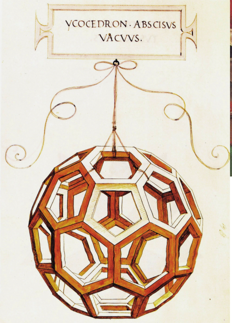

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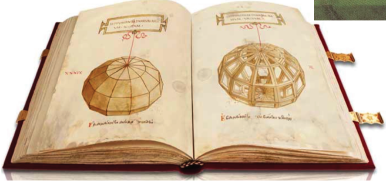

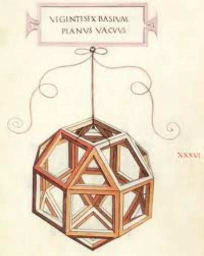

Back to Nuremberg

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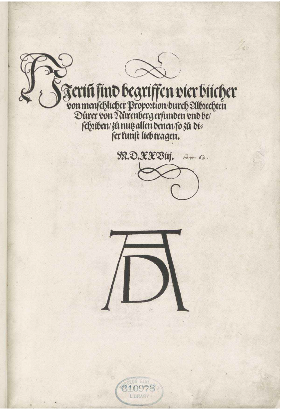
The Painter's Manual


Hierin sind begriffen vier bücher
von menschlicher Proportion/durch Albrechten
Dürer von Nürenberg erfunden vnd be/
schriben/zů nutz allen denen/so zů di/
ser kunst lieb tragen.

M.D.XXViij.

"The painter's manual: A manual of measurement of lines, areas, and solids by means of compass and ruler assembled by Albrecht Duerer for the use of all lovers of art with appropriate illustrations arranged to be printed in the year MDXXV [1525]."

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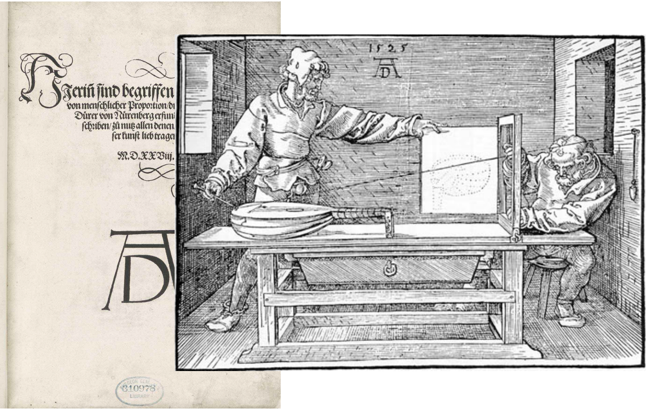

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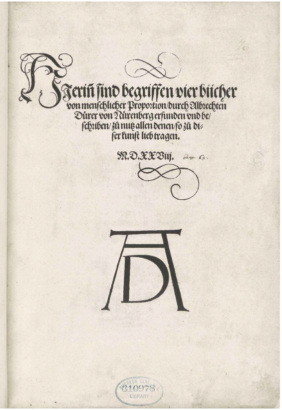
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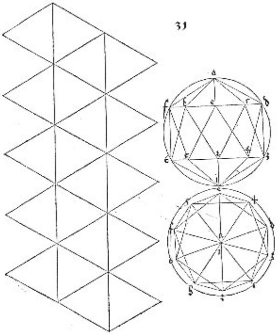

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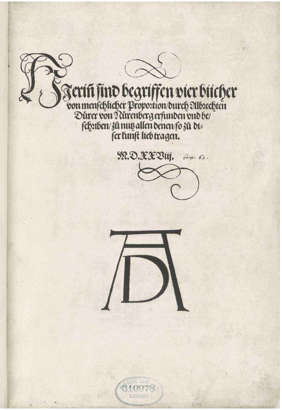

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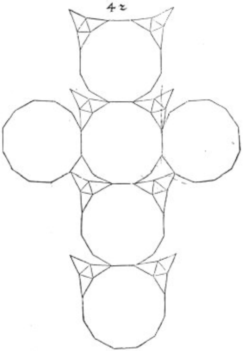

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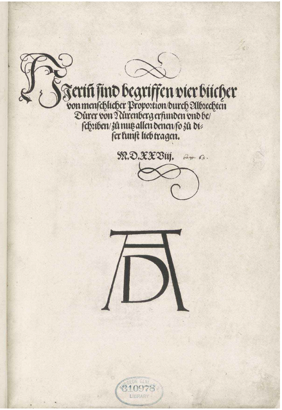
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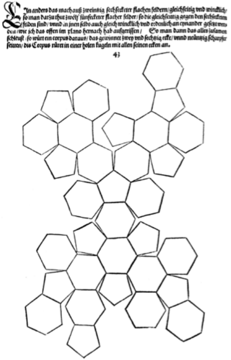

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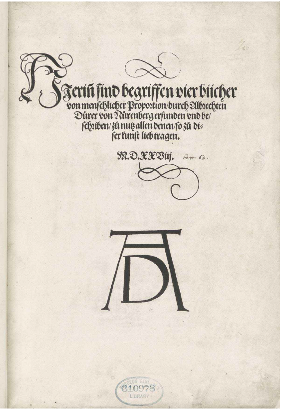

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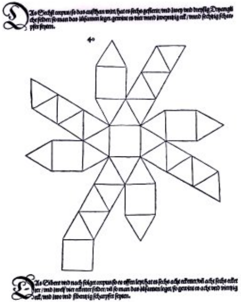

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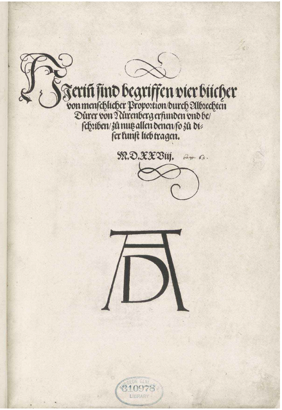
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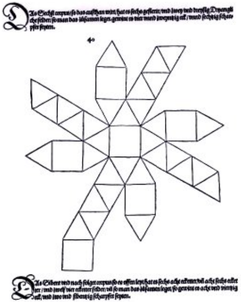

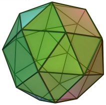

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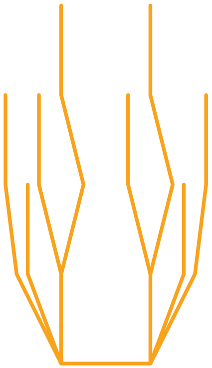
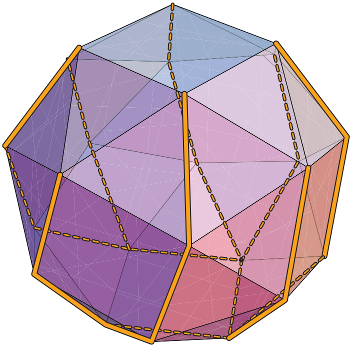
Dürer's cut tree of the snub cube





By the Gauss-Bonnet theorem, the cut set must be connected, and thus forms a spanning tree of the edge graph of the polyhedron.

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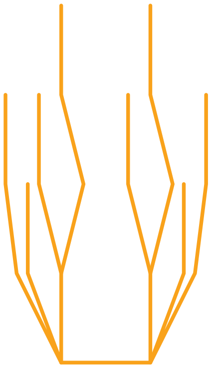

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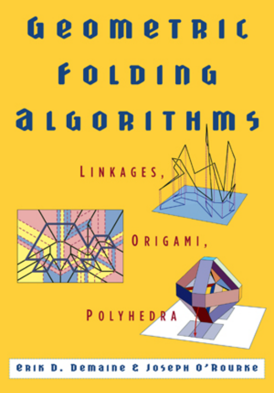

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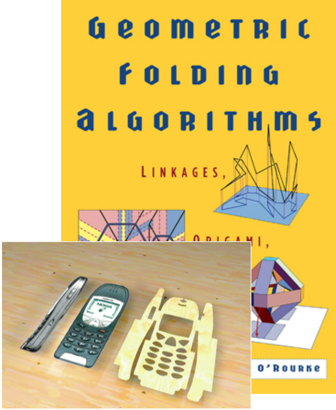

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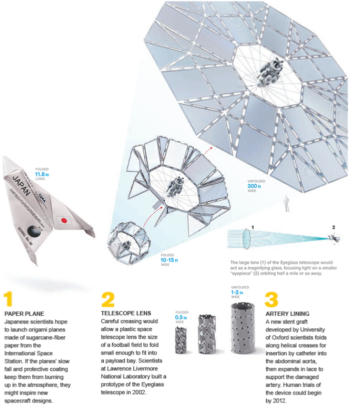

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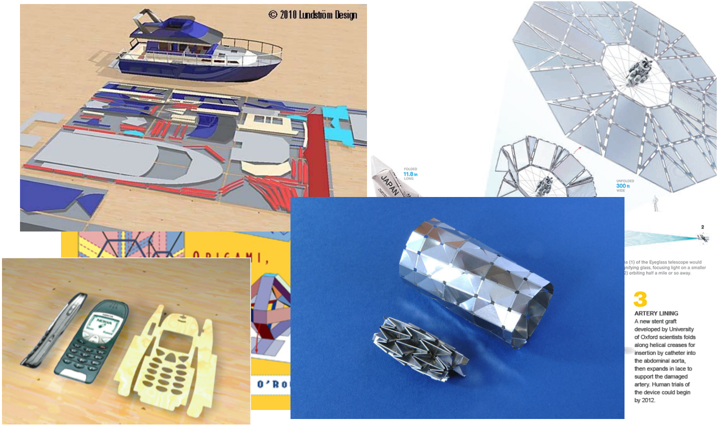

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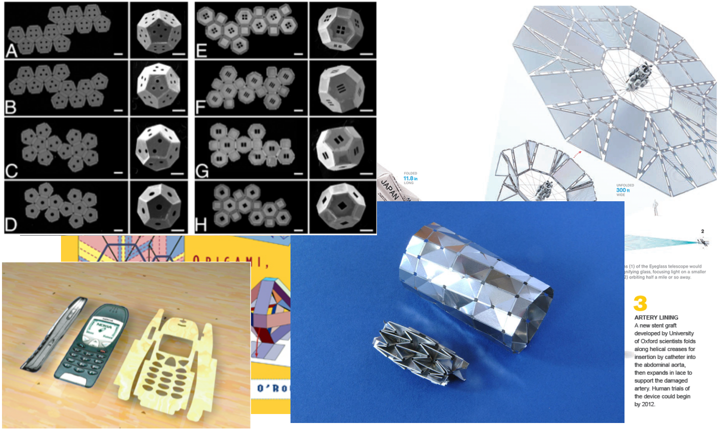

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Dürer's Problem

Problem

Can every convex polyhedron be cut along some spanning tree of its edges so that the resulting surface may be isometrically embedded in the plane?

- First explicitly formulated by G.C. Shephard in 1975.

- The assertion that the answer is yes has been dubbed "Dürer's Conjecture".

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Math. Proc. Camb. Phil. Soc. (1975), **78**, 389 389
With 12 text-figures
MPCPS 78–37
Printed in Great Britain

Convex polytopes with convex nets

By G. C. SHEPHARD
University of East Anglia, Norwich NR4 7TJ, England

(*Received* 10 *February* 1975)

- The assertion that the answer is yes has been dubbed "Dürer's Conjecture".

Dürer's Problem

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Can every convex polyhedron be cut along some spanning tree of its edges so that the resulting surface may be isometrically embedded in the plane?

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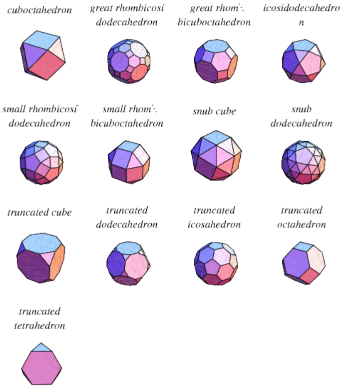
"Does every [convex] 3-polytope possess a net?"

- The assertion that the answer is yes has been dubbed "Dürer's Conjecture".

Evidence for Dürer's conjecture

- All known convex polyhedra are unfoldable, such as the Archimedean solids:



- The conjecture has been tested successfully in thousands (perhaps even millions) of cases.

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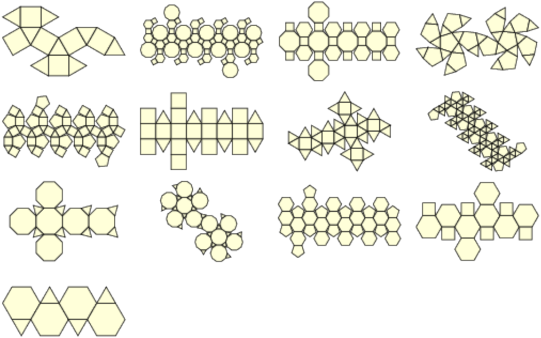

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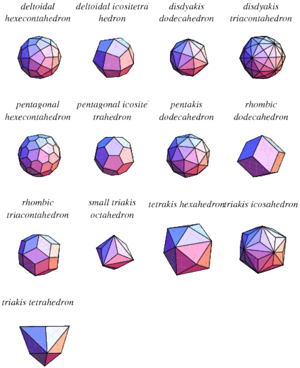
Evidence for Dürer's conjecture

- All known convex polyhedra are unfoldable, and the Archimedean duals:



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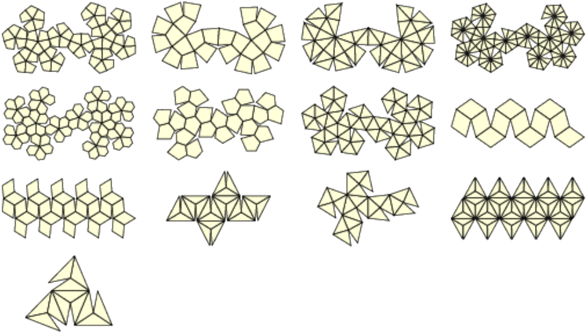

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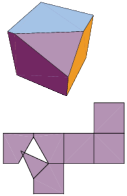
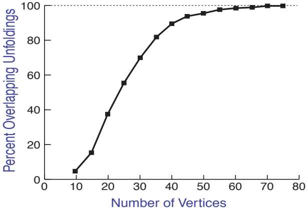
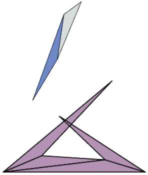
Evidence against Dürer's conjecture

- The probability that a random unfolding of a generic convex polyhedron is not simple approaches 1 as the number of vertices grow [O'Rourke & Schevon, 1987].



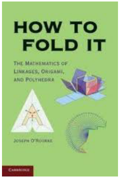


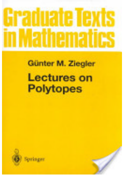


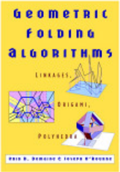
- Every known algorithm for simple unfoldings fails [Schlickenrieder, 1997].

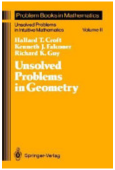
Dürer's problem is wide open

Although Dürer's problem has been well-publicized,

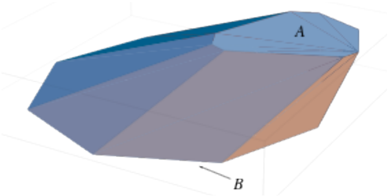






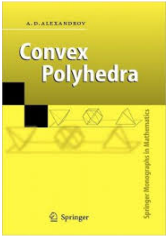


it is not even known if *prismatoids* are unfoldable.



Nor is it known if the vertex neighborhood of every face of a convex polyhedron is unfoldable.

A. D. Alexandrov (1912-1999)



Since shortest arcs issuing from the same point never meet again, after making cuts we obtain a (geodesic) polygon Q homeomorphic to a disk. The interior of Q contains no vertices. Hence, in accordance with Theorem 5 of Section 1.8, the diagonals split Q into triangles each of which is developable on a plane.

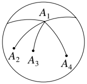


Fig. 88(a)

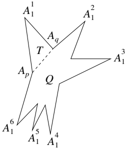


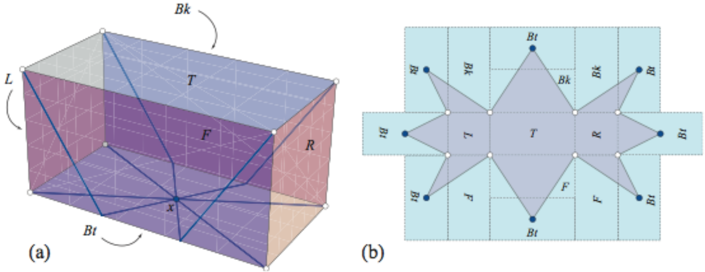
Fig. 88(b)

However, we wish to prove that Q splits by diagonals into triangles so that no two vertices of any triangle in the resultant development are glued together, i.e., correspond to the same vertex of the development.

The polygon Q has vertices of two types: $v-1$ vertices correspond to the vertices $A_2, A_3, \ldots, A_v$ of the development, while the other $v-1$ vertices correspond to the single vertex A_1 (Fig. 88(b)). These vertices $A_1^1, A_1^2, \ldots, A_1^{v-1}$, called vertices of the *second type*, alternate on the perimeter of Q. Since only vertices of the second type correspond to the same vertex of the development, we must prove the following:

The polygon Q may be split by diagonals into triangles in such a way that at most one vertex of each triangle is a vertex of the second type.

Example of Alexandrov's unfolding

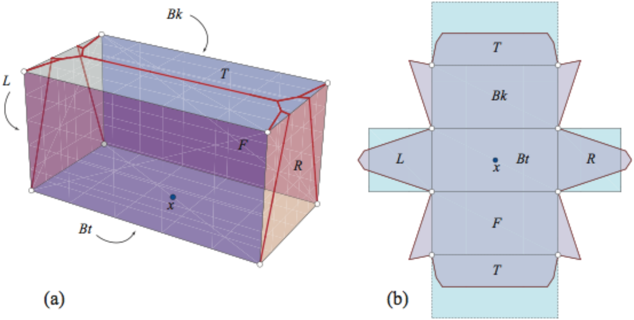


(a) (b)

[O'Rourke, 2008]

Cut along geodesics which connect a non-vertex point to all vertices.

Another method for generating simple general unfoldings

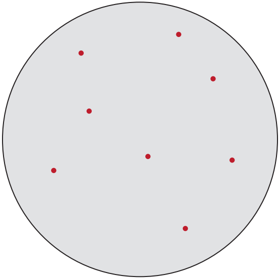


[O'Rourke, 2008]

Cut along the cut locus of a generic point

Why is Dürer's problem hard?

The problem with edge unfoldings is that we do not have an intrinsic characterization for an edge of a convex polyhedron.



An edge is a geodesic between a pair of vertices, but which pair?

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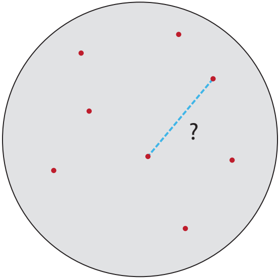

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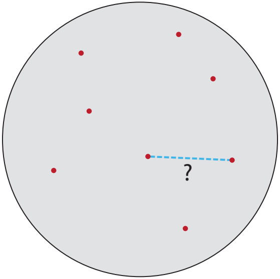

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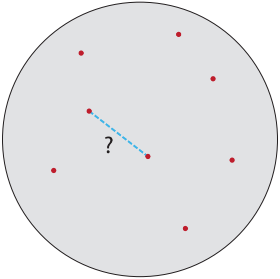
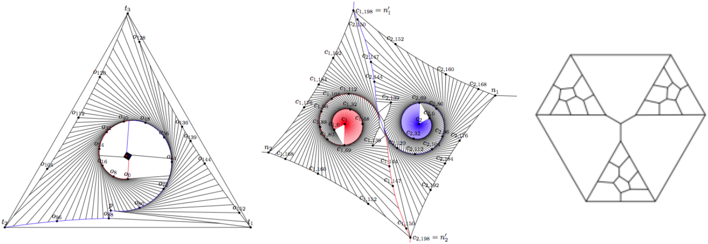


An edge is a geodesic between a pair of vertices, but which pair?

Tarasov's paper

The edge graph of a convex polyhedron P is not the only graph in P whose vertices are vertices of P, whose edges are geodesics, and whose faces are convex.

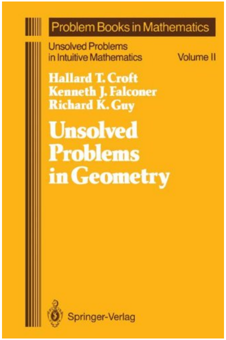


Tarasov [2008] has claimed that there exists a convex polyhedron with a convex partition which does not admit any simple unfolding. His example has 19008 vertices.

The question of Croft, Falconer, Guy

Question (CFG, 1991)

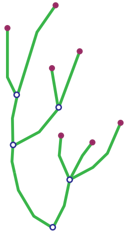
Is every convex polyhedron combinatorially equivalent to an unfoldable one?



The answer is yes.

Monotone trees

A *cut tree* T in a convex polyhedron P is a (polygonal) tree which includes all the vertices of P, and each of its leaves is a vertex of P. Suppose P has a unique top vertex and a unique bottom vertex with respect to some direction u.

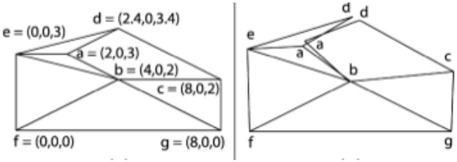


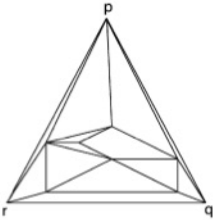
Then T is *monotone* with respect to u if the height function $\langle \cdot, u \rangle$ is decreasing on each branch of T which connects a leaf of T to the bottom vertex of P.

The steepest descent algorithm

One way to generate monotone trees is via the steepest descent algorithm.

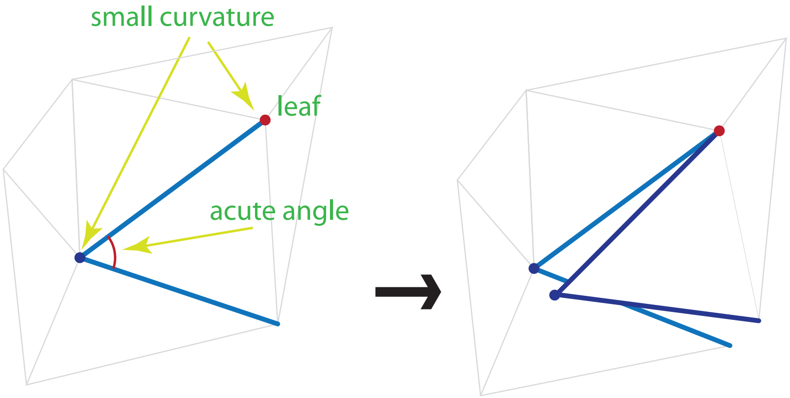
Schlickenrieder had conjectured that every convex polyhedron has at least one steepest descent edge tree which generates a simple unfolding, and had tested it successfully in thousands of cases.





But Lucier [2009] found a counterexample.

Recipe for constructing non-simple unfoldings



Affine stretching
(how to eliminate the acute angles of a monotone tree)

For $\lambda > 0$, the (normalized) *affine stretching* parallel to a direction u is the linear transformation $A_\lambda \colon \mathbf{R}^3 \to \mathbf{R}^3$ given by

$$A_\lambda(p) := \frac{1}{\lambda}\big(\, p + (\lambda - 1)\langle p, u\rangle u \,\big).$$

If $u = (0, 0, 1)$, then $A_\lambda(x, y, z) = (x/\lambda, y/\lambda, z)$. So A_λ makes convex polyhedra arbitrarily "thin" or "needle-shaped" for large λ.

For any $X \subset \mathbf{R}^3$, set

$$X^\lambda := A_\lambda(X).$$

Main result

Theorem (——, 2013)

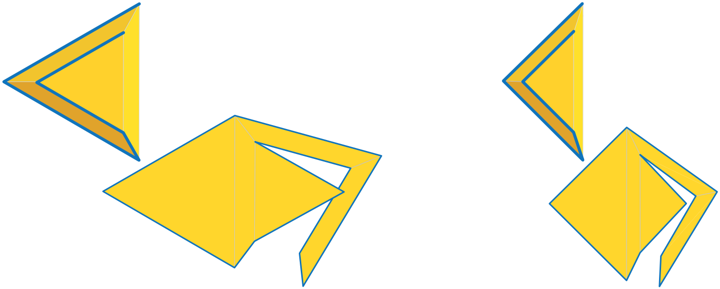
Let u be a direction with respect to which P is in general position, and T be a cut tree which is monotone with respect to u. Then the unfolding of P^λ generated by T^λ is simple for sufficiently large λ.

For almost every direction $u \in \mathbf{S}^2$, P admits a monotone cut tree which is composed of the edges of P. So:

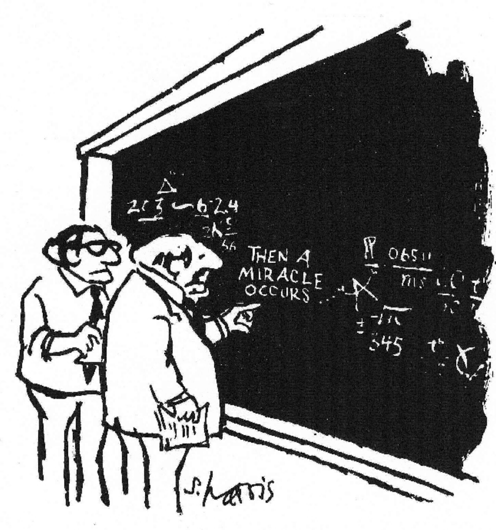
Corollary

Every convex polyhedron becomes unfoldable after an affine stretching in almost any direction.

Example



"Then a miracle happens"



"I think you should be more explicit here in step two."

An REU student, AJ Friend, verified the theorem experimentally before it was proved

The main idea of the proof

Induction on the number of branches of the cut tree.

Outline of the proof

1. Criteria for embeddedness of immersed disks
2. The tracing path of a tree
3. Mixed developments
4. Structure of piecewise monotone trees
5. Affine developments of piecewise monotone paths
6. Induction on the number of leaves

1. Criteria for embeddedness of immersed disks

Cutting P along T yields a topological disk P_T which is locally isometric to the plane and thus admits an isometric immersion $P_T \to \mathbf{R}^2$, which is the *unfolding* generated by T.

We say that the image $\overline{P_T}$ is *simple* if the unfolding map is one-to-one.

Lemma (———, *Topology*, 2002)

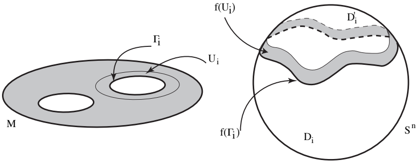
Let $f \colon M^n \to \mathbf{S}^n$, $n \geqslant 2$, be an immersion, where M is compact and connected. If f is one-to-one on each component of ∂M, then f is an embedding.

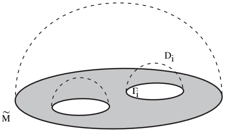
So $\overline{P_T}$ is simple if and only if $\overline{\partial P_T}$ is simple.

1. Criteria for embeddedness of immersed disks

Proof.

Extend $f\colon M \to \mathbf{S}^n$ to an immersion $\widetilde{f}\colon \widetilde{M} \to \mathbf{S}^n$, where $\widetilde{M}$ is a closed manifold containing M.





□

1. Criteria for embeddedness of immersed disks

Theorem

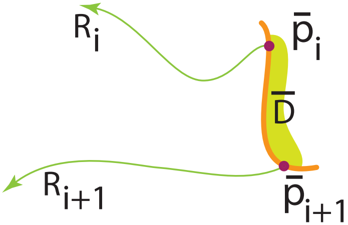
Let $D \to \mathbf{R}^2$ be an immersion. Suppose there are $k \geqslant 2$ points $p_i \in \partial D$, $i \in Z_k$, such that $\overline{p_i p_{i+1}}$ is simple and rays R_i emanating from $\overline{p}_i$ such that

(i) $R_i \cap R_{i+1} = \emptyset$,

(ii) $R_i \cap \overline{p_{i-1} p_{i+1}} = \{\overline{p}_i\}$,

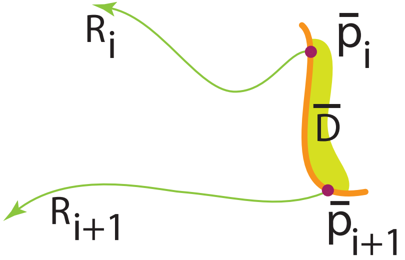
(iii) R_i *lies outside* $\overline{D}$ *near* $\overline{p}_i$.

Then $\overline{D}$ *is simple.*



1. Criteria for embeddedness of immersed disks

Proof:



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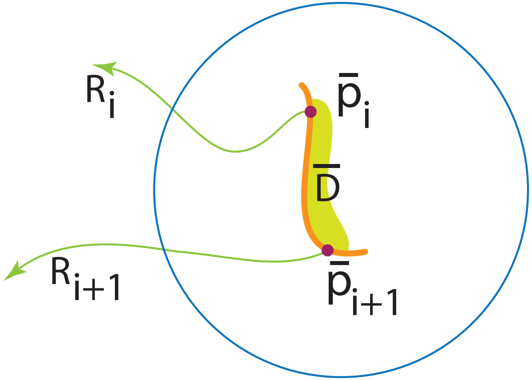

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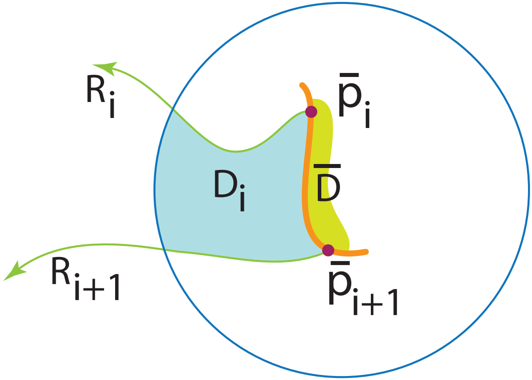
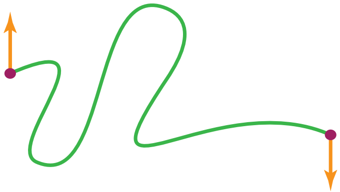


1. Criteria for embeddedness of immersed disks

A simple curve segment in $\mathbf{R}^2$ is *weakly monotone* if it may be extended to an unbounded simple curve by attaching vertical rays to its end points.



1. Criteria for embeddedness of immersed disks

Corollary

Let $D \to \mathbf{R}^2$ be an immersion. Suppose there is a pair of points p_0, p_1 in ∂D such that $\overline{p_0 p_1}$ and $\overline{p_1 p_0}$ are weakly monotone. Then $\overline{D}$ is simple.

1. Criteria for embeddedness of immersed disks

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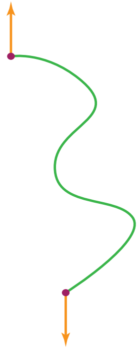

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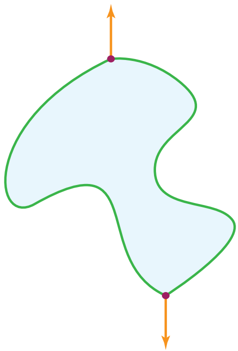


2. The tracing path of a cut tree

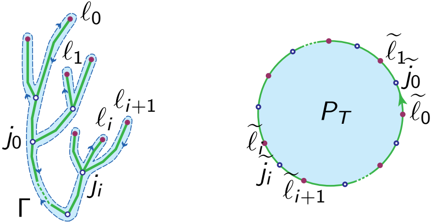
There is a canonical projection

$$P_T \to P.$$

Restricting this mapping to the boundary of P_T generates

$$\partial P_T \to T$$

which gives us the tracing pathm, say Γ, of T.



The boundary of the unfolding $\overline{P}_T$ coincides with the *development* of a path Γ in P which traces T.

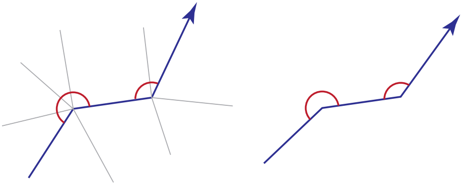
$$\overline{\partial P_T} = \overline{\Gamma}$$

2. The tracing path of a cut tree

A *path* is a sequence of oriented line segments such that the final point of each segment coincides with the initial point of the following segment.

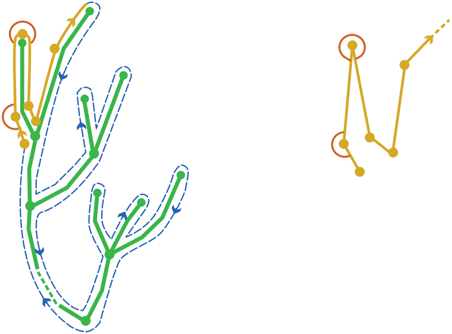
$$\Gamma = [v_0, \ldots, v_k] := (v_0 v_1, \ldots, v_{k-1} v_k).$$

A (left) *development* of a path Γ in P is a path $\overline{\Gamma}$ in $\mathbf{R}^2$ with the same edge lengths and left angles.



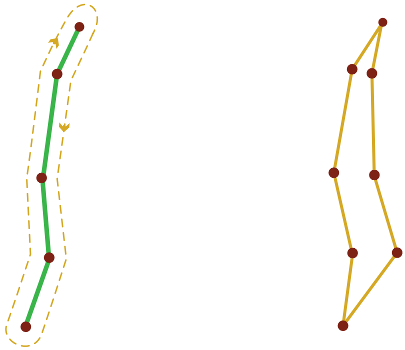
2. The tracing path of a cut tree

So all we need is to show that the development of the trace path is simple



2. The tracing path of a cut tree

In particular if the trace path has only one branch it follows quickly that the corresponding unfolding is simple (after an affine stretching):



Now toward the inductive step ...

3. Mixed developments

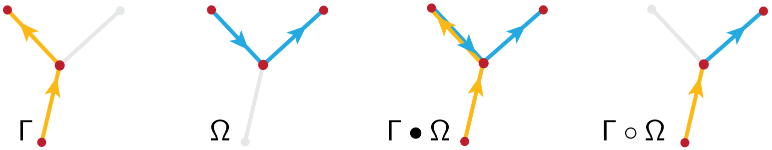
Let $\Gamma = [\gamma_0, \ldots, \gamma_k]$ and $\Omega = [\omega_0, \ldots, \omega_\ell]$, where $\gamma_k = \omega_0$. The *concatenation* of these paths is given by

$$\Gamma \bullet \Omega := [\gamma_0, \ldots, \gamma_k, \omega_1, \ldots, \omega_\ell],$$

while their *composition* is defined as

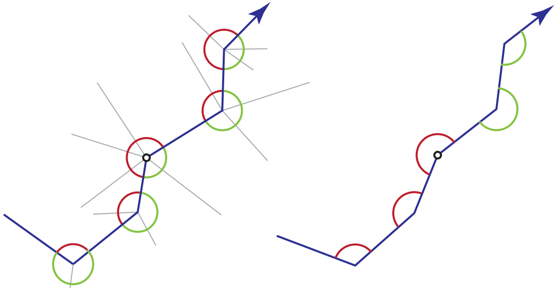
$$\Gamma \circ \Omega := [\gamma_0, \ldots, \gamma_{k-m}, \omega_{m+1}, \ldots, \omega_\ell],$$

where m is the largest integer such that $\gamma_{k-i} = \omega_i$ for $0 \leqslant i \leqslant m$.

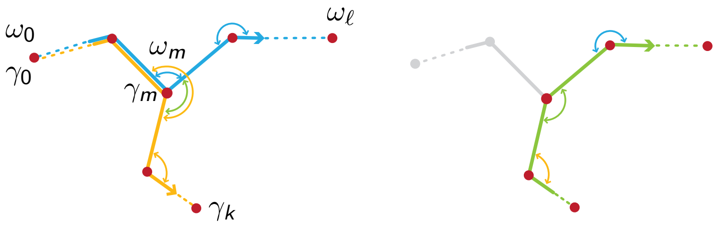


3. Mixed developments

Developing the composition of a pair of paths leads to the notion of *mixed* developments.



3. Mixed developments



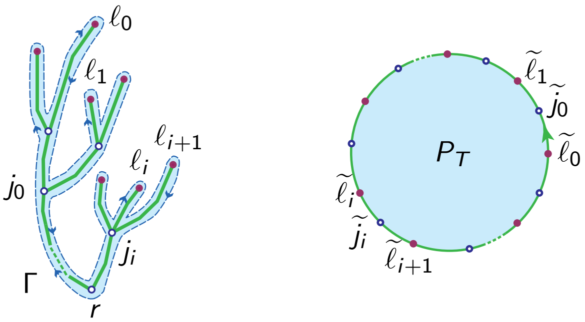
Proposition

Let $\Gamma = [\gamma_0, \ldots, \gamma_k]$, $\Omega = [\omega_0, \ldots, \omega_\ell]$ be a pair of paths in P such that $\gamma_i = \omega_i$ for $i = 0, \ldots, m < \ell$. Further suppose that either $m = k$, or else ω_{m+1} lies strictly to the left of $[\gamma_{m-1}, \gamma_m, \gamma_{m+1}]$. Then

$$(\overline{\Gamma})^{-1} \circ \overline{\Omega} \equiv \overline{(\Gamma^{-1} \circ \Omega)}_{\gamma_m},$$

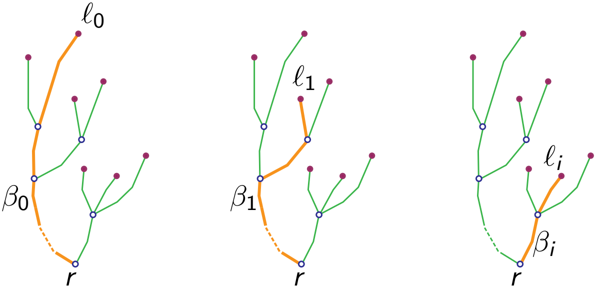
provided that $\overline{\Gamma}$ and $\overline{\Omega}$ have the same initial conditions.

4. Structure of monotone trees: leaves and junctures



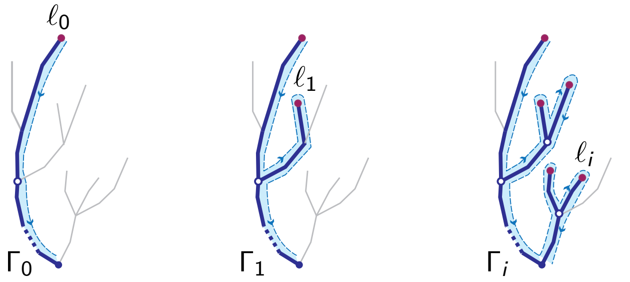
$$\Gamma = \ell_0 j_0 \bullet j_0 \ell_1 \bullet \cdots \bullet \ell_{k-1} j_{k-1} \bullet j_{k-1} \ell_0$$

4. Structure of monotone trees: branches



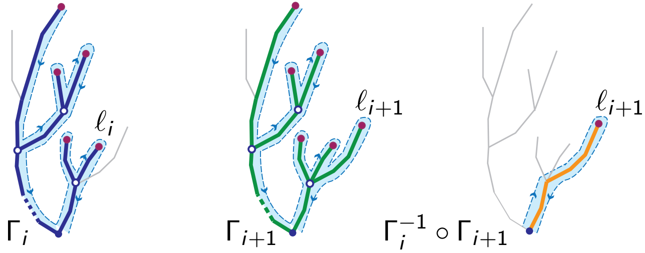
$$\beta_i := (\ell_i r)_T$$

4. Structure of monotone trees: the paths Γ_i



$$\Gamma_i := (\ell_0 \ell_i)_\Gamma \bullet \beta_i$$

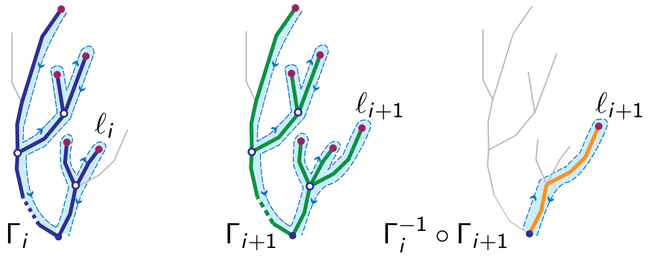
4. Structure of monotone trees: relation between β_i and Γ_i



Lemma

For $0 \leqslant i \leqslant k-2$, $\Gamma_i^{-1} \circ \Gamma_{i+1} = \beta_{i+1}^{-1} \bullet \beta_{i+1}$.

4. Structure of monotone trees: relation between β_i and Γ_i

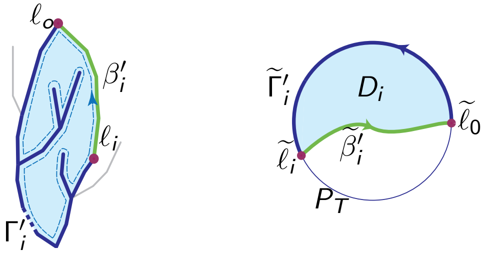


Proof.

$$\begin{aligned}
\Gamma_i^{-1} \circ \Gamma_{i+1} &= \left((\ell_0 j_i)_\Gamma \bullet (j_i r)_T\right)^{-1} \circ \left((\ell_0 \ell_{i+1})_\Gamma \bullet \beta_{i+1}\right) \\
&= (r j_i)_T \bullet (j_i \ell_0)_{\Gamma^{-1}} \circ (\ell_0 j_i)_\Gamma \bullet (j_i \ell_{i+1})_\Gamma \bullet \beta_{i+1} \\
&= (r j_i)_T \bullet (j_i \ell_{i+1})_T \bullet \beta_{i+1} \\
&= (r \ell_{i+1})_T \bullet \beta_{i+1} \\
&= \beta_{i+1}^{-1} \bullet \beta_{i+1}.
\end{aligned}$$

□

4. Structure of monotone trees: β'_l and Γ'_i



Proposition

Each leaf ℓ_i of a monotone cut tree T may be connected to the top leaf ℓ_0 of T via a monotone path β'_i in P, which intersects T only at its end points.

$$\Gamma'_i := \begin{cases} (\ell_0 \ell_i)_\Gamma \bullet \beta'_i, & 1 \leqslant i \leqslant k-1; \\ \Gamma, & i = k. \end{cases}$$

5. Affine developments of piecewise monotone paths

Proposition

Let Γ be a piecewise monotone path in P and $\overline{\Gamma}$ be a mixed development of Γ. Then each subpath of $\overline{\Gamma}^{\lambda}$ which corresponds to a positively (resp. negatively) monotone subpath of Γ^{λ} will be positively (resp. negatively) monotone.

5. Affine developments of piecewise monotone paths

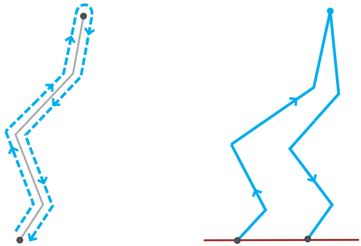
Let $D\Gamma := \Gamma \bullet \Gamma^{-1}$.

Proposition

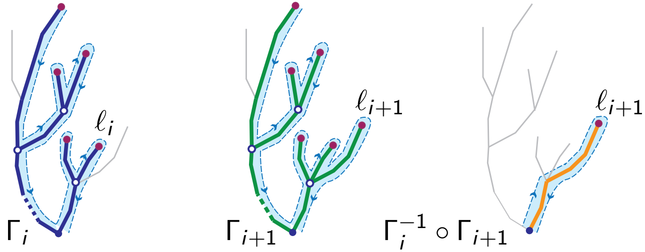
Let $\Gamma = [\gamma_0, \ldots, \gamma_k]$ be a monotone path in P ending at a vertex of P other than the top or bottom vertex, and $0 \leqslant \ell < k$. Then, for sufficiently large λ:

(i) $(\overline{D\Gamma})_{\gamma^\ell}$ *is simple.*

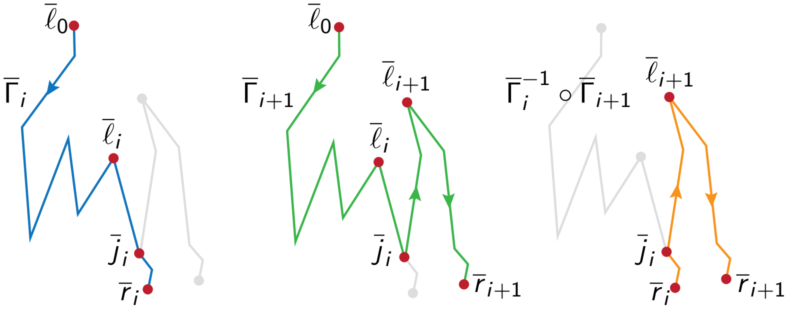
(ii) *The line which passes through the end points of $(\overline{D\Gamma})_{\gamma^\ell}$ intersects it at no other point, and is almost orthogonal to it.*



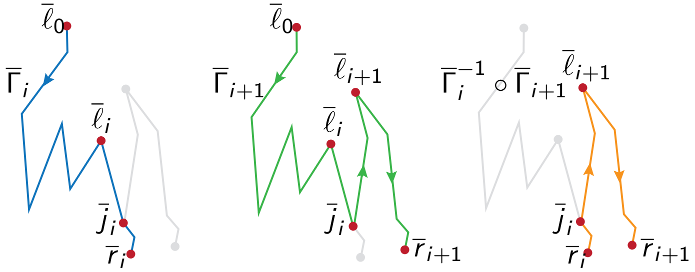
6. Induction on the number of leaves



6. Induction on the number of leaves



6. Induction on the number of leaves



Fix λ so large that:

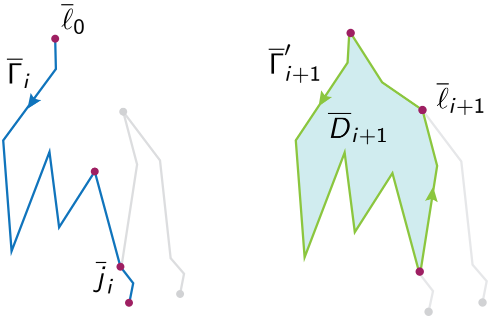
(i) For each positively (resp. negatively) monotone subpath of Γ_i or Γ'_i the corresponding subpath of $\overline{\Gamma}_i^\lambda$ or $(\overline{\Gamma'_i})^\lambda$ is positively (resp. negatively) monotone.

(ii) $(\overline{\Gamma}_i^\lambda)^{-1} \circ \overline{\Gamma}_{i+1}^\lambda$ is simple and lies on one side of the line L^λ passing through its end points. Furthermore, L^λ is not vertical.

6. Induction on the number of leaves

Lemma

For $0 \leqslant i \leqslant k-1$, if $\overline{\Gamma}_i$ is weakly monotone, then $\overline{\Gamma'}_{i+1}$ is simple.

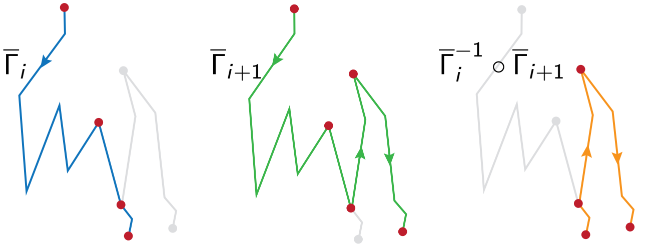


But $\Gamma'_k = \Gamma$. Thus it remains to show that $\overline{\Gamma}_{k-1}$ is weakly monotone ($\overline{\Gamma}_0 = \overline{\beta}_0$ is already weakly monotone by our choice of λ).

6. Induction on the number of leaves

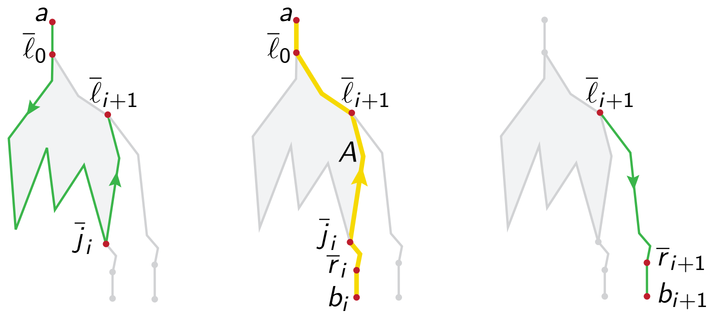
Lemma

For $0 \leqslant i \leqslant k-2$, if $\overline{\Gamma}_i$ is weakly monotone, then so is $\overline{\Gamma}_{i+1}$.



6. Induction on the number of leaves

Proof.



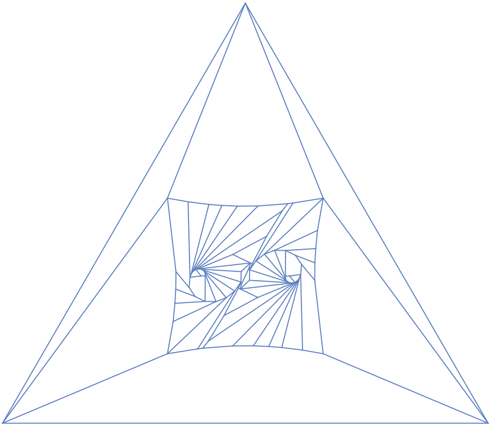
$$A := b_i\overline{r}_i \bullet (\overline{r_i j_i})_{\overline{\Gamma}_i^{-1}} \bullet (\overline{j_i \ell_0})_{\overline{\Gamma'}_{i+1}} \bullet \overline{\ell}_0 a,$$

□

A more recent result (joint work with Nicholas Barvinok)

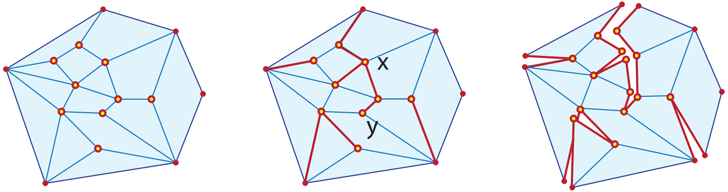
Theorem

There exists a convex polyhedron K with 384 vertices and a pseudo-edge graph with respect to which K is not unfoldable.



Lemma (Pogorelov)

Let P be a convex polygon, p_i, $i = 1, \dots, n$, be points in the interior of P, and $\beta_i > 0$ with $\sum_i \beta_i < 2\pi$. Then there exists a unique convex cap C over P with interior vertices v_i such that $\pi(v_i) = p_i$, and $k(v_i) = \beta_i$



We say that x is an *ancestor* of y, or y is a *descendant* of x, and write $x \preceq y$ if x lies in the (uniqe path) which connects x to the root of its tree.

The *center of rotation* of x is the center of mass of its descendant vertices with respects to the weights α_i:

$$c_x := \alpha_x^{-1} \sum_{i \succeq x} \alpha_i p_i, \quad \text{where} \quad \alpha_x := \sum_{i \succeq x} \alpha_i.$$

Tarasov's Monotonicity Condition

Every interior vertex p_i of the cut forrest F has a unique adjacent vertex p_i^* in F which is its *parent* or first strict ancestor which is a vertex.

A cut forest F is *monotone*, if for every interior vertex p_i of F we have

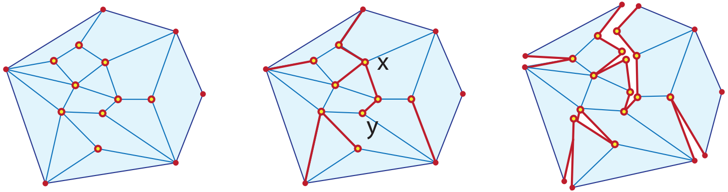
$$\langle p_i^* - p_i, p_i - c_i \rangle \geqslant 0, \quad \text{where} \quad c_i := c_{p_i}.$$

So, if $p_i \neq c_i$ and $0 \leqslant \angle p_i^* p_i c_i \leqslant \pi$ denotes the angle between the vectors $p_i^* - p_i$ and $c_i - p_i$, then we have

$$\angle p_i^* p_i c_i \geqslant \pi/2.$$

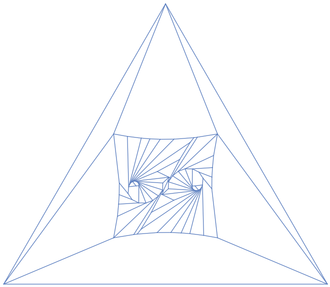
Theorem

If a cut forrest of a convex partition of a convex polygon is not monotone, then the corresponding convex cap over it cannaot have any simple unfolding with respect to that cut forrest.



Theorem (Simplifies an earlier construction by Tarasov)

There exists a weighted convex partition of the equilateral triangle, with 84 interior vertices, which does not admit any monotone cut forrest!

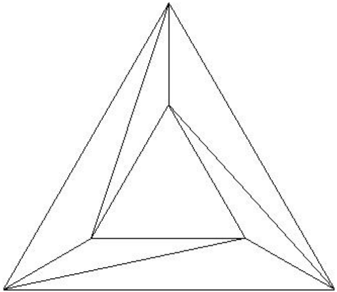


Taking 4 copies of shallows convex caps over this triangle, and arranging them on the faces of a regular tetrahedron, yields the a convex polyhderon with no simple pseudo-edge unfolding.

Can the previous partition, or something similar, be used to construct an actual counterexample to Durrer's conjecture?

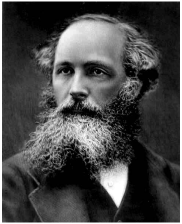
That would depend on wether a convex partition can be lifted to a convex cap *faithfuly*, i.e., so that edges of the partition lift to edges of the cap?

Not every convex partition can be lifted faithfully



This partition has no faithful lifting.

Maxwell-Cremona Criterion for lifting

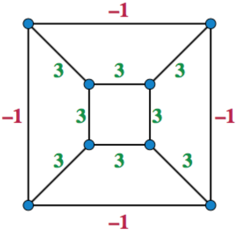
A convex partition may be lifted faithfully provided that it admits an *equllibrium stress*.

A convex partition may be lifted faithfully provided that it admits an *equllibrium stress*.

A *stress* on a planar graph is an assignment of scalars ω_{ij} to its edges. A stress is in *equilibrium* if for each vertex v_i,

$$\sum_{j=1}^{deg(v_i)} \omega_{ij}(v_i - v_j) = 0$$

where v_j are adjacent vertices of v_i.



So, can one construct a counterexample to Durer's conjecture?

If one exists, then one way to try to find it would be to construct a convex partition of the equilateral triangle, with distribution of weights α_i at its vertices, and ω_{ij} along its edges, such that

1. α_i do not satisfy Tarasov's monotonicity criterion for any cut forrest
2. ω_{ij} satisfy the Maxwell-Cremona lifting criterion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Thanks!