Mathematics 1501 Hour Examination
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Directions: Do all problems. Show your work and justify your answers. This is a closed book examination. Make sure your copy of the examination has four (4) distinct pages; put your name and your recitation leader's name on each page.

1 (20) a. Find all values of x which satisfy $x(x^2-25) \geq 0$.

$x(x^2-25) = (x+5)x(x-5)$ If $x<-5$, this product is negative. If $-5<x<0$, it is positive. If $0<x<5$, it is negative. If $5<x$, it is positive. Thus

$x(x^2-25) \geq 0 \iff -5 < x < 0$ or $5 < x$

b. Find all values of x which satisfy $|3x-6|<3$.

$|3x-6|<3 \iff 3|x-2|<3 \iff |x-2|<1 \iff 1<x<3$

2 (10) Show that the function $f(x) = x\cos(\frac{1}{1+x^2})$ is continuous at zero.

$f(0)=0$. Also $|x\cos(\frac{1}{1+x^2})| \leq |x|$, so $-|x| \leq x\cos(\frac{1}{1+x^2}) \leq |x|$

Since $|x| \to 0$ as $x \to 0$, the Pinching Theorem shows that

$\lim_{x\to 0} x\cos(\frac{1}{1+x^2}) = 0 = f(0)$. Thus f is continuous at zero

or

$\cos t$ is defined and continuous for all t, and $\frac{1}{1+x^2}$ is defined and continuous for all x. Thus $\cos(\frac{1}{1+x^2})$ is continuous for all x. Since x is continuous for all x, the product $x\cos(\frac{1}{1+x^2})$ is continuous at zero

3 (30) Which of the following limits exist? If the limit does not exist, give a reason why not. If it does exist, evaluate it. You may use the fact (established in class) that

$\lim_{x \to 0} \frac{\sin x}{x} = 1$.

a. $\lim_{x \to 0} \frac{x}{\tan(x)} = \lim_{x \to 0} \frac{x}{\frac{\sin x}{\cos x}} = \lim_{x \to 0} \frac{x}{\sin x} \cos x = \left(\frac{1}{1}\right) 1 = 1$

b. $\lim_{x \to 3} \frac{2x^2 - 18}{(3-x)x} = \lim_{x \to 3} \frac{2(x-3)(x+3)}{(3-x)x} = \lim_{x \to 3} \frac{2(x+3)}{x} = \frac{2(6)}{3} = 4$

c. $\lim_{x \to 0} \frac{\sqrt{x^2+36} - 6}{3x^2} = \lim_{x \to 0} \left(\frac{\sqrt{x^2+36} - 6}{3x^2}\right)\left(\frac{\sqrt{x^2+36} + 6}{\sqrt{x^2+36} + 6}\right)$

$= \lim_{x \to 0} \frac{x^2 + 36 - 36}{3x^2\left(\sqrt{x^2+36} + 6\right)} = \lim_{x \to 0} \frac{x^2}{3x^2\left(\sqrt{x^2+36} + 6\right)}$

$= \lim_{x \to 0} \frac{1}{3\left(\sqrt{x^2+36} + 6\right)} = \frac{1}{3\left(\sqrt{36} + 6\right)} = \frac{1}{3(6+6)} = \frac{1}{36}$

Name Key Recitation leader [redacted]

4. (30) a. What is the domain of the function $f(x)=\sqrt{x-2}+\frac{1}{x^2-16}$?

Domain of $\sqrt{x-2}$ is $\{x : x \geq 2\}$

Domain of $\frac{1}{x^2-16} = \{x : |x| \neq 4\}$

Domain of sum is $\{x : x \geq 2 \text{ and } x \neq 4\} = [2, 4) \cup (4, +\infty)$

b. Let $f(x)=\frac{1}{x^2}$ and $g(x)=\sqrt{x^2-4}$. Give explicit formulas for $f \circ g$ and for $g \circ f$.

$f \circ g(x) = \frac{1}{\left(\sqrt{x^2-4}\right)^2} = \frac{1}{x^2-4}$

$g \circ f(x) = \sqrt{\left(\frac{1}{x^2}\right)^2 - 4} = \sqrt{\frac{1}{x^4} - 4}$

c. Find the domain of the function $f \circ g$ from part b above.

Domain of $g = (-\infty, -2] \cup [2, +\infty)$ i.e. $|x| \geq 2$

Domain of $f = \{x : x \neq 0\}$

Domain of $f \circ g = \{x : |x| \geq 2 \text{ and } \sqrt{x^2-4} \neq 0\}$

$= \{x : |x| \geq 2 \text{ and } x^2 \neq 4\}$

$= \{x : |x| > 2\} = (-\infty, -2) \cup (2, +\infty)$

5. (10) Use an epsilon-delta argument (that is, use the definition of limit) to show that $\lim_{x \to 1} 3x - 5 = -2$.

Let $\varepsilon > 0$.

We want $|3x - 5 - (-2)| < \varepsilon$, i.e. $|3x - 3| < \varepsilon$

Now $|3x - 3| < \varepsilon \iff 3|x-1| < \varepsilon \iff |x-1| < \varepsilon/3$

Choose $\delta = \varepsilon/3$.

If $0 < |x-1| < \delta$, then $|x-1| < \varepsilon/3$, so

$3|x-1| < \varepsilon$, so $|3x-3| < \varepsilon$, so

$|3x - 5 - (-2)| < \varepsilon$.