NAME: ______________________ Rec. leader: ______________

Mathematics 1501 Hour Examination
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Directions: Do all problems. Show your work and justify your answers. Calculators are not allowed, and this is a closed book examination. You are allowed one prepared sheet of paper. Make sure your name is on all four pages of your examination.

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1 (40) For each of the following sequences $f(x)$, either compute the limit of the sequence or show that the limit does not exist.

a. $x_n = \frac{n-2n^2}{5n^2+2n+1}$ $= \frac{\frac{1}{n} - 2}{5 + \frac{2}{n} + \frac{1}{n^2}} \longrightarrow -\frac{2}{5}$

b. $x_n = (\frac{2n-6}{10+2n})^n$ $= \frac{\left(1 - \frac{3}{n}\right)^n}{\left(1 + \frac{5}{n}\right)^n} \longrightarrow \frac{e^{-3}}{e^5} = e^{-8}$

c. $x_n = (-1)^n (1 + \frac{3}{n})^n$ $x_{2n} = \left(1 + \frac{3}{2n}\right)^{2n} \to e^3$ but $x_{2n+1} = -\left(1 + \frac{3}{2n+1}\right)^{2n+1} \longrightarrow -e^{+3}$. Since $e^3 \neq -e^{+3}$, this limit does not exist

1 (continued) For each of the following sequences, either compute the limit of the sequence or show that the limit does not exist.

d. $x_n = \sqrt{\dfrac{n^2+1}{16n^2-4n}}$ $= \sqrt{\dfrac{1+\frac{1}{n^2}}{16-\frac{4}{n}}} \longrightarrow \sqrt{\dfrac{1}{16}} = \dfrac{1}{4}$

Since the square root function is continuous

2 (10) Suppose that a function f is differentiable everywhere and satisfies $f(n) = 0$ for every integer n. Suppose also that $f'(x) \geq 0$ for all x. What can you say about f and why?

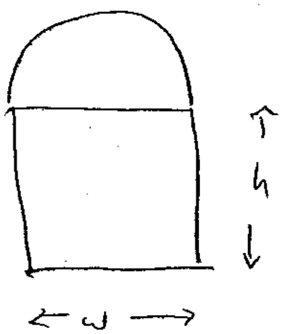
We must have $f(x) = 0$ for all x.

For suppose x is given and $n \leq x \leq n+1$

Then since $f'(x) \geq 0$ for all x, f is non-decreasing. Thus $0 = f(n) \leq f(x) \leq f(n+1) = 0$

Thus $f(x) = 0$

3. (20) A Norman window has the shape of a rectangle surmounted by a semicircle. (Thus the diameter of the semicircle is equal to the width of the rectangle.) If the perimeter of the window is 30 feet, find the dimensions of the window so that the greatest possible amount of light is admitted.



$$\text{Area} = \frac{1}{2}\pi\left(\frac{w}{2}\right)^2 + wh = \frac{\pi}{8}w^2 + wh$$

$$\text{Perimeter} = 2h + w + \frac{1}{2}\pi w$$

$$= 2h + \left(1 + \frac{\pi}{2}\right)w$$

Thus $30 = 2h + \left(1+\frac{\pi}{2}\right)w$, so $h = 15 - \left(\frac{1}{2} + \frac{\pi}{4}\right)w$

Thus $\text{Area} = A(w) = \frac{\pi}{8}w^2 + w\left(15 - \left(\frac{1}{2} + \frac{\pi}{4}\right)w\right)$

$$= \frac{\pi}{8}w^2 - \left(\frac{1}{2} + \frac{\pi}{4}\right)w^2 + 15w = \left(-\frac{1}{2} - \frac{\pi}{8}\right)w^2 + 15w$$

$$A'(w) = 0 \iff \left(-1 - \frac{\pi}{4}\right)w + 15 = 0 \iff w = \frac{15}{1 + \frac{\pi}{4}}$$

Since $A''(w) = -\left(1 + \frac{\pi}{4}\right) < 0$, this critical point gives a local maximum. Since it is the only critical point, it represents an absolute maximum. $h = 15 - \left(\frac{1}{2} + \frac{\pi}{4}\right)\left(\frac{15}{1 + \frac{\pi}{4}}\right)$

4 (30) Let $f(x) = x^4 - 5x^2 + 4$. Note that $f(x) = 0$ if and only if $x = -2$, $x = -1$, $x = 1$ or $x = 2$.

a. Find and classify (as a local maximum, a local minimum, or neither) all of the critical points of f.

$0 = f'(x) = 4x^3 - 10x = x(4x^2 - 10) \iff x = 0, \; x = \frac{\sqrt{5}}{\sqrt{2}}$ or $x = -\frac{\sqrt{5}}{\sqrt{2}}$

$f''(x) = 12x^2 - 10$ is negative at $x = 0$ and positive at $x = \pm\frac{\sqrt{5}}{\sqrt{2}}$. Thus $x = 0$ gives a local max

$x = \pm\frac{\sqrt{5}}{\sqrt{2}}$ gives two local minima

b. Find all the inflection points of f (i.e., all the points where the concavity of the graph of f changes).

$x = \pm\sqrt{\frac{10}{12}} = \pm\sqrt{\frac{5}{6}}$ are the roots of $12x^2 - 10$

These are both inflection points

c. Sketch the graph of f, showing the local maxima and minima and the concavity properties of the graph.