Mathematics 1501 Hour Examination

W. L. Green

December 2, 2005

Directions: Do all problems. Show your work, and justify your answers. Calculators are not allowed, and this is a closed book examination. You are allowed one prepared sheet. Please put your name and your recitation leader's name on each page of your examination.

Page	
1	
2	
3	
4	
Total	

1. (32) Calculate each of the following derivatives and integrals.

a. $\frac{d}{dx}\{\ln[4(x+1)^3]\}$ $= \frac{d}{dx}\left[\ln 4 + 3\ln(x+1)\right]$

$= \frac{3}{x+1}$

b. $\int_0^{\pi/6} \tan x\,dx$ $= \left[\ln|\sec x|\right]_0^{\pi/6}$

$= \ln\left|\sec\left(\frac{\pi}{6}\right)\right| - \ln|\sec(0)| = \ln\left(\frac{2}{\sqrt{3}}\right) - \ln(1) = \ln\left(\frac{2}{\sqrt{3}}\right)$

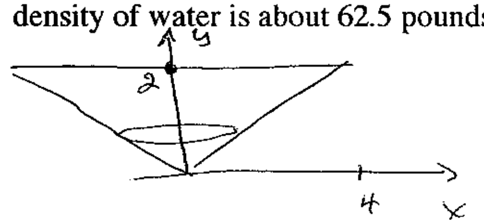
$= \ln 2 - \ln\sqrt{3}$

c. $\frac{d}{dx}(2^{5x})$ $= \frac{d}{dx}\left[e^{5x\ln 2}\right] = (5\ln 2)e^{5x\ln 2} = (5\ln 2)2^{5x}$

d. $\int \frac{4}{1+x^2}dx$ $= 4\int \frac{1}{1+x^2}dx = 4\,\text{Arctan}\,x + C$

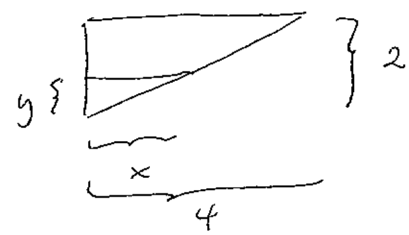
2. (24) A water tank is in the shape of a right circular cone with radius 4 ft. and height 2 ft., so that the cone can be pictured as the region generated by revolving the triangle with vertices (0,0), (0,2) and (4,2) about the y-axis. The tank is full of water.

a. How much work is done in pumping all of the water out of the tank? The weight density of water is about 62.5 pounds per cubic foot.



$0 \le y \le 2$

$A(y) = \pi x^2$



$\frac{y}{x} = \frac{2}{4}$, so $x = 2y$

$$\text{Work} = \int_0^2 (2-y)\left(62\tfrac{1}{2}\right)\pi x^2\, dy$$

$$= \int_0^2 \frac{125\pi}{2}(2-y)(2y)^2\, dy = 250\pi \int_0^2 2y^2 - y^3\, dy$$

$$= 250\pi\left[\frac{2}{3}y^3 - \frac{1}{4}y^4\right]_0^2 = 250\pi\left(\frac{16}{3} - 4\right) = 250\pi\left(\frac{16}{3} - \frac{12}{3}\right)$$

$$= \frac{1000\pi}{3}$$

b. Suppose that only the water in the top foot of the tank is pumped out. How much work is done?

$$\text{Work} = \int_1^2 (2-y)\left(\frac{125}{2}\pi\right)(2y)^2\, dy = 250\pi \int_1^2 2y^2 - y^3\, dy$$

$$= 250\pi\left[\frac{2}{3}y^3 - \frac{1}{4}y^4\right]_1^2 = \frac{1000\pi}{3} - 250\pi\left(\frac{2}{3} - \frac{1}{4}\right)$$

$$= \frac{1000\pi}{3} - 250\pi\left(\frac{8}{12} - \frac{3}{12}\right) = \frac{1000\pi}{3} - \left(\frac{5}{12}\right)250\pi = \frac{\pi}{36}(10{,}750)$$

3. (20) At noon on October 31, a sample of a radioactive isotope contains 10 micrograms. At noon on November 20, it contains 8 micrograms.

a. How much of the isotope was there at noon on November 10?

$$y(t) = 10e^{kt}$$

$$8 = y(20) = 10e^{20k}$$

$$\frac{4}{5} = e^{20k}$$

$$\ln\left(\frac{4}{5}\right) = 20k$$

$$k = \frac{1}{20}\ln\left(\frac{4}{5}\right) = \frac{\ln 4 - \ln 5}{20}$$

$$y(10) = 10e^{\frac{10}{20}\ln\left(\frac{4}{5}\right)} = 10\left(\frac{4}{5}\right)^{10/20} = 10\sqrt{\frac{4}{5}} = 4\sqrt{5} \text{ micrograms}$$

b. What is the half-life of this isotope?

$$\frac{1}{2}(10) = (10)e^{kt_h}$$

$$\frac{1}{2} = e^{kt_h}$$

$$-\ln 2 = kt_h$$

$$t_h = \frac{k}{-\ln 2} = \frac{\ln 4 - \ln 5}{-20\ln 2} = \frac{\ln 5 - \ln 4}{20\ln 2}$$

4. (12) Simplify each of the following.

a. $e^{3\ln 2}$ $= 2^3 = 8$

b. $\ln(4x) - \ln(2)$ $= \ln\left(\frac{4x}{2}\right) = \ln 2x = \ln 2 + \ln x$

c. $2^{\log_3 9}$ $= 2^2 = 4$

d. $\log_2 \sqrt{32}$ $= \log_2\left[(2^5)^{1/2}\right] = \log_2(2^{5/2}) = \frac{5}{2}$

5. (8) Find the integral $\int xe^{2x}dx$.

$$\int xe^{2x}dx = \int \frac{u}{2} e^u \frac{du}{2}$$

$u = 2x$

$du = 2dx$

$dx = \frac{du}{2}$

$$= \frac{1}{4}\int ue^u du = \frac{1}{4}\int u\, d(e^u)$$

$$= \frac{1}{4}\left[ue^u - \int e^u du\right] = \frac{1}{4}\left[ue^u - e^u\right] + C$$

$$= \frac{1}{4}\left[2xe^{2x} - e^{2x}\right] + C = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C$$