

Geometric and Harmonic Means of Operators
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This is a lecture about a small part of operator theory. To be specific, it is about generalizing to positive operators the familiar arithmetic, geometric and harmonic means of positive numbers. The key tools in this generalization are spectral theory, which we shall briefly describe below, and a theorem of R. G. Douglas, which is widely known but ought to be even more widely known. (See below for a statement of this theorem.)

Recall that if a and b are positive numbers, then the arithmetic, geometric and harmonic means are given respectively by the three expressions

$$\frac{a+b}{2}$$
$$\sqrt{ab}$$
$$2(a^{-1} + b^{-1})^{-1} = \frac{2ab}{a+b}.$$

Related to the harmonic mean is the parallel sum, given by

$$(a^{-1} + b^{-1})^{-1} = \frac{1}{\frac{1}{a} + \frac{1}{b}} = \frac{ab}{a+b}.$$

Standard notations for the parallel sum and the geometric mean of a and b are

$$a : b = (a^{-1} + b^{-1})^{-1}$$

and

$$a \# b = \sqrt{ab}.$$

You may recall from elementary physics that the parallel sum $a:b$ represents the resistance of a pair of circuit elements, of resistance a and b respectively, connected in parallel. Since the harmonic mean of a and b is just twice the parallel sum, the study of harmonic means can be accomplished by focusing on the parallel sum whenever it is convenient to do so.

Any decent piece of mathematics should consist of a more than just a few definitions, so here is a theorem, called the arithmetic-geometric-harmonic mean inequality (AGH inequality):

Theorem: For all positive a and b , we have $2(a : b) \leq \sqrt{ab} \leq \frac{a+b}{2}$.

The proof of the AGH inequality is an easy exercise in high school algebra.

Now suppose that A and B are positive operators on a Hilbert space H . (In this talk all operators are assumed to be bounded and linear and all Hilbert spaces are assumed complex.) We would like to generalize the three means and the parallel sum to such pairs A and B , and to prove as many theorems (like the AGH inequality) as possible. So how do we do this?

The expression $\frac{A+B}{2}$ is well defined and seems appropriate for the arithmetic mean, so we will use it as is. However, the parallel sum and the geometric mean do present some problems when we try to generalize the formulas above to positive operators. First we have the problem that A or B might be non-zero and yet not be invertible. This difficulty is relatively easy to overcome; one can for example add a small positive multiple of the identity to each of A and B (this makes them invertible) and take a limit to get a definition which coincides with $(A^{-1} + B^{-1})^{-1}$ when A and B actually are invertible. (There is some checking to be done to see that this works, but it does work, and it is not deep.) However, the expression \sqrt{AB} simply will not do for the geometric mean: unless A and B commute, the product AB is not a positive operator, and it is not clear that this square root even exists.

So how do we define the geometric mean of A and B ? The answer is that we use spectral theory. To see how this works, let us begin by observing that if a and b are non-negative real-valued functions (on some set X whose exact identity need not yet concern us), then the formulas

$$\frac{a+b}{2}$$

$$\sqrt{ab}$$

$$\frac{2ab}{a+b}$$

make perfectly good sense and define non-negative functions on X . To be fair, we need to concede that there might be a problem if a and b are both zero at some point of x , but a straightforward argument from advanced calculus shows that if we define $f(x,y)$ to be zero on both the x - and y -axes, then the function

$$f(x,y) = \frac{xy}{x+y}.$$

has a continuous extension to the closed first quadrant, and it follows that $\frac{2ab}{a+b}$ has a well defined and non-negative continuous extension (which is zero when either of a and b is zero) for all non-negative a and b .

Thus, if we could somehow regard A and B as functions, then we would be able to define our means and might be able to prove some theorems. This is actually asking a little too much, but it turns out that we can associate to A and B another pair of positive operators which will serve as satisfactory substitutes. We need the result of Douglas.

Theorem: Let S and T be bounded linear operators on H . Then the following conditions are equivalent.

- (a) There exists a bounded linear C such that $S = TC$
- (b) There exists $\lambda \geq 0$ such that $SS^* \leq \lambda TT^*$
- (c) $R(S) \subseteq R(T)$, where $R(W)$ denotes the range of W .

Moreover if these conditions are satisfied, then there is a unique operator C_0 which satisfies (a) and has range contained in the orthogonal complement of the kernel of T . For this unique C_0 we have $\text{kernel}(C_0) = \text{kernel}(S)$, and we have $\|C_0\|$ equal to the square root of the least λ which satisfies (b).

The proof of this theorem is not particularly difficult. It uses the closed graph theorem to show that (c) implies (a).

Returning now to our positive A and B we argue as follows. Since A and B are positive, we have

$$A \leq A + B \text{ and } B \leq A + B,$$

which means that

$$A^{1/2}A^{1/2} \leq (A + B)^{1/2}(A + B)^{1/2} \text{ and } B^{1/2}B^{1/2} \leq (A + B)^{1/2}(A + B)^{1/2}.$$

For simplicity, let us assume that the kernel of $(A + B)^{1/2}$ is zero, and hence that the orthogonal projection of H onto the closure of the range of $A + B$ is the identity operator I . (This is not a serious restriction.) By Douglas' result (with $S = A^{1/2}$, and $T = B^{1/2}$), there exist unique operators C and D such that

$$\begin{aligned} A^{1/2} &= (A + B)^{1/2} C \text{ and} \\ B^{1/2} &= (A + B)^{1/2} D. \end{aligned}$$

This means that

$$\begin{aligned} A + B &= A^{1/2}A^{1/2} + B^{1/2}B^{1/2} \\ &= (A + B)^{1/2} CC^*(A + B)^{1/2} + (A + B)^{1/2} DD^*(A + B)^{1/2} \\ &= (A + B)^{1/2} [CC^* + DD^*] (A + B)^{1/2} \end{aligned}$$

so that

$$(A + B)^{1/2}(A + B)^{1/2} = A + B = (A + B)^{1/2} [CC^* + DD^*] (A + B)^{1/2}$$

Since $(A + B)^{1/2}$ is injective (its kernel is zero), we have

$$[CC^* + DD^*](A + B)^{1/2} = (A + B)^{1/2}$$

Taking adjoints gives

$$(A + B)^{1/2} [CC^* + DD^*] = (A + B)^{1/2}$$

From the injectivity again we get

$$CC^* + DD^* = I.$$

Note that this last implies that $CC^*DD^* = DD^*CC^*$.

Here finally we are in a position to use some spectral theory. The *spectrum* of an operator T is the set of all those scalars λ for which $T - \lambda I$ is not invertible. (Think eigenvalue.) It is always a non-empty compact subset of the complex plane. One can show that an operator is positive if and only if it is self-adjoint and has a spectrum consisting entirely of non-negative real numbers. The basic idea of spectral theory is that if T is positive (actually, normal will do), then for any continuous function f on the spectrum of T , there is an operator $f(T)$ which behaves among operators the way f does among functions. To be more precise, there is an isometric adjoint-preserving isomorphism between the closed $*$ -algebra generated by T and the algebra of all continuous functions on T . (This is one version of the spectral theorem.) Moreover, this isomorphism preserves the ordering of operators.

Now consider $T = CC^*$. To write T as $f(T)$ we need to use the identity function for f , so T corresponds to the function $f(t) = t$ on the spectrum of T . We have $DD^* = I - CC^*$, so the function that corresponds to DD^* is $1 - \text{identity}$, i.e., the function which takes t to $1 - t$. Now the fact that $CC^* \leq I$ implies that the spectrum of CC^* is a subset of the closed unit interval $[0,1]$. For t in this interval we have

$$2t(1-t) \leq \sqrt{t(1-t)} \leq \frac{1}{2} = \frac{t + (1-t)}{2}.$$

(This is just the AGH inequality for numbers.) By spectral theory we get the same inequality for the corresponding operators:

$$2CC^*(I - CC^*) \leq \sqrt{CC^*(I - CC^*)} \leq \frac{I}{2} = \frac{CC^* + (I - CC^*)}{2},$$

i.e.,

$$2CC^*DD^* \leq \sqrt{CC^*DD^*} \leq \frac{I}{2} = \frac{CC^* + DD^*}{2}$$

Multiplying on both sides by $(A + B)^{1/2}$ gives

$$(A+B)^{1/2}(2CC^*DD^*)(A+B)^{1/2} \leq (A+B)^{1/2}\sqrt{CC^*DD^*}(A+B)^{1/2} \leq$$

$$\frac{A+B}{2} = (A+B)^{1/2} \frac{CC^*+DD^*}{2} (A+B)^{1/2}$$

It now seems clear that the harmonic, geometric and arithmetic means of A and B should be defined as the appropriate parts of this last set of inequalities. If we do this, we not only have our definitions, but we have actually established the AGH for operators. (Note too that this avoids the problem of the non-invertibility of A or B.) For additional results in this vein, please buy my book (when I am done writing it). Thank you.

WLG, 10/4/05