Mathematics 4317 Midterm Examination – Sept.27, 2006

Directions: Do all four of the problems below. Show your work, and justify your answers and assertions. ("Justify your answer" means give a proof or a counterexample.) This is a closed book examination, and calculators are allowed. Throughout this examination, the symbol "**R**" will denote the real number system.

1a) (15) Use the supremum property of **R** to prove the following: every non-empty subset S of **R** that is bounded below has a greatest lower bound.
b) (10) Show that if a subset of **R** has a least upper bound, then the least upper bound is unique.

2a) (10) It can be shown that the equation

$$\langle (x_1, y_1), (x_2, y_2) \rangle = x_1 x_2 + 2 y_1 y_2$$

defines an inner product $\langle \ \rangle$ on $\mathbf{R}^2$. (You may take this for granted.) Use this fact to show that for all real numbers x_1, y_1, x_2 and y_2 we have

$$(x_1 x_2 + 2 y_1 y_2)^2 \le (x_1{}^2 + 2 y_1{}^2)(x_2{}^2 + 2 y_2{}^2).$$

b) (15) Let H denote the upper half-plane $\{(x,y)$ in $\mathbf{R}^2 : y > 0 \}$. Show that H is an open subset of the plane $\mathbf{R}^2$.

3. (25) Let X be a set. For each subset S of X, define a function χ_S with domain X by

$$\chi_S(x) = 1 \text{ if } x \in S \text{ and } \chi_S(x) = 0 \text{ if } x \notin S .$$

Let Φ be the function defined by $\Phi(S) = \chi_S$. Show that Φ is a bijection from the set of all subsets of X onto the set of all functions from X to $\{0,1\}$.

4. Let S be a subset of $\mathbf{R}^p$, and suppose S has no cluster points.
a) (5) Must S be bounded? Justify your answer.
b) (5) Must S be closed? Justify your answer.
c) (15) Prove the following result (which we proved in class): Let $\{K_n\}$ be a nested sequence of non-empty closed bounded subsets of $\mathbf{R}^p$ (so that $K_n \supseteq K_{n+1}$ for every $n \ge 1$). Then the intersection $\bigcap_{n=1}^{+\infty} K_n$ of the sets $\{K_n\}$ is not empty.

1a) Let $T = \{x \in \mathbb{R} : -x \in S\}$. Then $l \leq s$ for all $s \in S$ implies that $-s \leq -l$ for all $s \in S$, i.e. that $t \leq -l$ for all $t \in T$. Thus T is non-empty (since S is non-empty) and bounded above. Let u be the least upper bound for T. Then $-x \leq u$ for all $x \in S$, so $-u \leq x$ for all $x \in S$, so $-u$ is a lower bound for S. Suppose l is any other lower bound for S. Then as above, $-l$ is an upper bound for T, so $u \leq -l$ (since u is the least upper bound for T), so $l \leq -u$. Thus $-u$ is the ~~least~~ greatest lower bound for S.

b) Let u_1 and u_2 be least upper bounds for S. Since u_1 is a least upper bound and u_2 is an upper bound, $u_1 \leq u_2$. Similarly $u_2 \leq u_1$, so $u_1 = u_2$.

2a) Let $|||(x,y)||| = \langle (x,y),(x,y)\rangle^{1/2}$. By Cauchy-Schwarz, we have

$$|x_1 x_2 + 2y_1 y_2| = |\langle (x_1,y_1),(x_2,y_2)\rangle| \leq |||(x_1,y_1)|||\ |||(x_2,y_2)|||, \text{ i.e.}$$

$$|x_1 x_2 + 2y_1 y_2| \leq (x_1^2 + 2y_1^2)^{1/2}(x_2^2+2y_2^2)^{1/2}.$$ Now square both sides.

b) The boundary points of the complement of H are the points on the x-axis. Since they are all contained in the complement, this complement is closed. Thus H is open.

3 ~~Suppose~~ S_1 and S_2 are subsets of X and $S_1 \neq S_2$. If $x \in S_1$ but $x \notin S_2$, then $\chi_{S_1}(x) = 1$ and $\chi_{S_2}(x) = 0$, so $\chi_{S_1} \neq \chi_{S_2}$. If $x \in S_2$ ~~then $\chi_S(x) = 0$~~ but $x \notin S_1$, then $\chi_{S_1}(x) = 0$ and $\chi_{S_2}(x) = 1$, so $\chi_{S_1} \neq \chi_{S_2}$. Thus if $S_1 \neq S_2$, then $\chi_{S_1} \neq \chi_{S_2}$, so Φ is 1-1. Let f be a function from X into $\{0,1\}$. Let $S = \{x : f(x) = 1\}$. Then for $x \in X - S$, $f(x) = 0$. Thus $f = \chi_S$. Thus Φ is onto.

4a) No: the integers are not bounded in $\mathbb{R}$ and have no cluster point.

b) Yes: S contains all of its cluster points, and so is closed.

c) See your text for a proof.

Z.J.R 9/27/06