Mathematics 4431 Final Examination – December 15, 2006

Directions: Do all problems. Show your work, and justify your answers and assertions. ("Justify your answer" means give a proof or a counterexample.) This is a closed book examination, and calculators are allowed. Throughout this examination, the symbol "**R**" will denote the real number system; unless otherwise specified, we will assume **R** has the usual topology. Please put your name on your bluebook.

1. (20) Let X be a topological space, and let p be a point of X.
a. Show that the intersection of any finite collection of neighborhoods of p is again a neighborhood of p.
b. Give an example of a topological space X and a point p in X such that closure of $\{p\}$ in X is different from $\{p\}$ (i.e, such that $\{p\}$ is not a closed subset of X).

2. (10) Let **Q** denote the set of all rational numbers, with the subspace (i.e., relative) topology from the usual topology on **R**. Show that if x is an element of **Q**, then the connected component of **Q** that contains x is $\{x\}$. [Hint: between any two distinct rationals in **R** there lies an irrational number.]

3. (10) Let $\{X_a\}$ be a collection (not necessarily countable) of compact Hausdorff spaces, and let X be the product of the collection $\{X_a\}$, with the product topology. Show that a subset S of X is compact (in the relative topology) if and only if it is closed in X.

4. (20) Let $\{x_n\}$ be a sequence in a metric space X.
a. Show that if p is a point of X, and if $\{x_n\}$ does *not* converge to p, then there exist a neighborhood U of p and a subsequence of $\{x_n\}$ that lies in the complement X – U of U.
b. Suppose X is compact, and suppose $\{x_n\}$ has exactly one cluster point in X. Show that $\{x_n\}$ converges.

5. (10) Suppose f is a uniformly continuous function from a metric space (X,d) into a complete metric space (Y,ρ). Show that if $\{x_n\}$ is a Cauchy sequence in (X,d), then $\{f(x_n)\}$ is a convergent sequence in (Y,ρ).

6. (30) For each of the following, either give an example of what is described below or give a reason why no such example can exist:
a. a totally ordered set that is not well ordered
b. a compact pseudometric space that is not Hausdorff
c. a closed subset of **R** (with the usual topology) that has no limit points
d. a continuous function on a compact space X into **R** that is not a closed function
e. a topological space with a countable base that is not separable
f. a net that is not a sequence.

1 a) Let $U_1, \dots, U_n$ be neighborhoods of p. Choose open sets $O_1, \dots, O_n$ with $p \in O_i \subseteq U_i$ for all i. Then $p \in \bigcap_{i=1}^{n} O_i \subseteq \bigcap_{i=1}^{n} U_n$, and $\bigcap_{i=1}^{n} O_i$ is open.

b) Let X be any set with at least two points and the trivial topology. Then for each p in X, the closure of $\{p\}$ is $X \neq \{p\}$.

2 Let C be the component of $\mathbb{Q}$ that contains x. Suppose C contains two distinct points y and z. We may assume $y < z$. Choose an irrational number v with $y < v < z$. Then $\{a \in \mathbb{Q} : a < v\}$ and $\{b \in \mathbb{Q} : v < b\}$ are non-empty open subsets of $\mathbb{Q}$, and their intersections with C disconnect C: $\{a \in C : a < v\}$ and $\{b \in C : v < b\}$ are open and closed and non-empty proper subsets of C.

3 Suppose S is compact in X. The product of Hausdorff spaces is Hausdorff, so S is a compact subset of a Hausdorff space X, so S is closed in X. Conversely, if S is closed in X, then S is a closed subspace of the product of compact spaces, which is compact by the Tychonoff Theorem, so S is itself compact.

4 a) Suppose $\{x_n\}$ does not converge to p. Then there exists a neighborhood U of p such that $\{x_n\}$ fails to lie eventually in U. Thus for each $m \geq 1$, there exists n_m with $n_m \geq m$ and $x_{n_m} \notin U$. Choose $n_1 \geq 1$ with this property. Supposing that we have chosen $n_1 < n_2 < \dots < n_m$ with $i \leq n_i$ for all i, ~~now choose n_{m+1}~~ and $x_{n_i} \notin U$ for all i, now choose n_{m+1} with $n_{m+1} \gneq \max\{n_1, n_2, \dots, n_m, m\}$ and $x_{n_{m+1}} \notin U$. By induction, we get a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with $x_{n_i} \in X - U$ for all i.

b) Let p be the unique cluster point of $\{x_n\}$ in X. Suppose $\{x_n\}$ does not converge to p. Then by part a), there exists $\{x_{n_i}\}$ and U with $x_{n_i} \in X - U$ for all i. We may assume U is an open neighborhood of p. Then $X - U$ is closed. Since X is compact, $X - U$ is compact. Thus $\{x_{n_i}\}$ has a cluster point z in $X - U$. But a cluster point of $\{x_{n_i}\}$ is also a cluster point of $\{x_n\}$, so we have reached a contradiction.

5 Let $\varepsilon > 0$. Choose $\delta > 0$ such that $d(x, y) < \delta$ implies $\rho(f(x), f(y)) < \varepsilon$. Now choose N so that $n \geq N$ and $m \geq N$ imply that $d(x_n, x_m) < \delta$. Then $\rho(f(x_n), f(x_m)) < \varepsilon$. Thus $\{f(x_n)\}$ is Cauchy. Since (Y, ρ) is complete, $\{f(x_n)\}$ is convergent.

6 a) $\mathbb{R}$ with the usual ordering

b) Let X be any set with at least two elements and give it the trivial pseudometric $d(x,x)=0$ and $d(x,y)=1$ for $x \neq y$.

c) the empty set

d) This cannot happen. If C is closed in X, then C is compact, so $f(C)$ is compact. Since $\mathbb{R}$ is Hausdorff, $f(C)$ is closed in $\mathbb{R}$.

e) This cannot happen. Let $\{B_i\}$ be a countable base for the topology, and choose $x_i \in B_i$. Then $\{x_i\}$ is a countable dense subset

f) Let $\mathcal{F}$ be the set of all finite subsets of $\mathbb{R}$. For each $F \in \mathcal{F}$, choose a point x_F in $\mathbb{R}$. Then $\{x_F\}_{F\in\mathcal{F}}$ is a net in $\mathbb{R}$, where $F_1 \prec F_2$ means $F_1 \subseteq F_2$. Clearly $\mathcal{F} \neq \mathbb{N}$.

Or

Let $\mathcal{D}$ be the collection of all open intervals in $\mathbb{R}$ of the form $(-\varepsilon, \varepsilon)$, where $\varepsilon > 0$. For each $i \in \mathcal{D}$, choose $x_i \in \mathcal{D}$. We order $\mathcal{D}$ by $(-\varepsilon,\varepsilon) \prec (-\delta,\delta) \iff \delta < \varepsilon$. Then $\{x_i\}_{i\in\mathcal{D}}$ is a net in $\mathbb{R}$ and $\{x_i\}$ converges to zero. Note that $\mathcal{D}$ is uncountable, so $\mathcal{D} \neq \mathbb{N}$.