## Order Relations

A *relation* R *between two sets* A *and* B is a subset of the Cartesian product A × B. If R is a relation between A and A, then R is said to be a relation *on* A (or *in* A). The set of all first members of a relation R is its *domain*, and the set of all second members is its *image*. If R is a relation between A and B and the domain of R is A, then R is said to be a relation *from* A *to* B. Observe that if A is empty, then the empty relation (i.e., the empty set) is the unique relation on A.

A relation R on a set X is a *partial order* if it is reflexive, anti-symmetric and transitive, i.e. if

a) for all x in X, $(x,x) \in R$;
b) for all x and y in X, $(x,y) \in R$ and $(y,x) \in R$ imply that $x = y$; and
c) for all x, y and z in X, $(x,y) \in R$ and $(y,z) \in R$ imply that $(x,z) \in R$.

When R is a partial order on X, we shall often for the sake of convenience write "$x \leq y$" to symbolize "$(x,y) \in R$", and "$x < y$" to symbolize "$(x,y) \in R$ and $x \neq y$" (i.e., to symbolize "$x \leq y$ and $x \neq y$"). Note that by transitivity and anti-symmetry, we have for any partial order that

d) $x \leq y$ and $y < z$ imply $x < z$, and $x < y$ and $y \leq z$ imply $x < z$.

For any partial order on X and any x and y in X, we have the first only, the second only, both, or neither of the two conditions "$x \leq y$" and "$y \leq x$". Thus by anti-symmetry, we see that exactly one of the four conditions

"$x < y$", "$y < x$", "$x = y$", and "x and y are not related"

holds.

A partial order $\leq$ on X is a *total order* if for every pair x and y in X we have either $x \leq y$ or $y \leq x$. (That is, a partial order $\leq$ on X is a total order if and only if every pair in X is related by $\leq$.) When $\leq$ is a total order on X, then for any pair x and y in X we have the usual trichotomy condition that exactly one of

"$x < y$" or "$y < x$" or "$x = y$"

holds.

In a partially ordered set X, an element a is a *maximum element* of X if $x \leq a$ for every x in X; an element a is a *maximal element* of X if there is no element x of X other than a itself such that $a \leq x$. *Minimum elements* and *minimal elements* are defined similarly. A maximum element or minimum element is (by anti-symmetry) always unique, but maximal and minimal elements are rarely unique in general partially ordered sets. (Two distinct maximal elements, or two distinct minimal elements, must be incomparable, by anti-symmetry.) Note that a subset of a partially ordered (resp., totally ordered) set is itself partially (resp., totally) ordered by the relation inherited from the larger set.

A partially ordered set X is *well ordered* if every non-empty subset of X has a minimum element (a minimum element for that subset). In this context, a minimum element is often called "a least element". Note that every well ordered set is totally ordered, and that if X is empty, then the unique (empty) ordering on X is a well ordering. If X is well ordered by $\leq$, then every subset of X is also well ordered by $\leq$.

A subset S of a well-ordered X is an *initial segment of* X if $s \in S$, $x \in X$, and $x \leq s$ imply that $x \in S$, i.e., if

$$\{x \in X : \text{there exists } s \text{ in } S \text{ with } x \leq s\} \subseteq S.$$

Note that X and the empty set are both initial segments of X for any well ordered X.

If A is a subset of a well-ordered set and $y \in A$, we shall denote the set $\{ x \in A : x < y \}$ by A(y). If $x \in A(y)$ and $z \leq x$ for some z in A, then $z \leq x < y$, so that $z \in A(y)$; thus we see that each A(y) is an initial segment of A. Conversely, we have the following result.

**Proposition**: Suppose S is an initial segment of a well-ordered set A. If $S \neq A$, then there exists y in A such that $S = A(y)$.

**Proof**: Let y be the minimum element of $A - S$. Suppose $x \in A(y)$, so that $x \in A$ and $x < y$; if x were not in S, then by minimality of y we would have $y \leq x$, contradicting $x < y$. Thus $A(y) \subseteq S$. Suppose conversely that $x \in S$. If we then had $y \leq x$, it would follow that $y \in S$, since S is an initial segment of A. But this would contradict the fact that y lies in $A - S$. Thus $x < y$ whenever $s \in S$, so that $S \subseteq A(y)$.

Observe that if $A(y) = A(z)$, then $z < y$ is false (otherwise $z \in A(y) = A(z)$, so that $z < z$), so that $y \leq z$; similarly $z \leq y$, so that $y = z$. This establishes the following strengthening of the proposition.

**Corollary**: Suppose S is an initial segment of a well-ordered set A. If $S \neq A$, then there exists a unique element y of A such that $S = A(y)$.

From the proof of the proposition, we see that the element y of this corollary is just the least element of $A - S$. In particular, for any non-empty well-ordered set A, the empty set is the initial segment A(y), where y is the least element of A.

**Proposition**: Let A be a subset of a well-ordered set, and let $y \in A$. Then $A(y) \cup \{y\}$ is an initial segment of A.

**Proof**: Let $x \in A(y) \cup \{y\}$, and suppose $z \leq x$. If $x \in A(y)$, then $z \leq x < y$, so that $z \in A(y)$, so that $z \in A(y) \cup \{y\}$. Suppose then that $x = y$. If $z = x$, then clearly $z \in A(y) \cup \{y\}$; if $z < x$, then $z < y$, so that $z \in A(y) \subseteq A(y) \cup \{y\}$. Thus in all cases, $z \in A(y) \cup \{y\}$, so $A(y) \cup \{y\}$ is an initial segment of A.

**Proposition**: Let A be a well-ordered set, and let **S** be a collection of initial segments of A.

Then the union B of the segments in **S** is again an initial segment of A.

**Proof**: Let b be an element of the union B, and let $x \in A$ with $x \leq b$. Then $b \in S$ for some S in **S**. Since S is an initial segment of A and $x \leq b$, we have $x \in S$, so x is an element of the union B.

**Theorem**: Let X be a partially ordered set, and suppose that every well-ordered subset of X (with the ordering inherited from X) has an upper bound in X. Then X has a maximal element.

**Remark**: Note that as the empty subset of X is well ordered, the hypotheses of the theorem imply that X is not empty.

**Proof of the Theorem**: Please see your text.

**Corollary (Zorn's Lemma)**: Let X be a partially ordered set, and suppose that every totally ordered subset of X (with the ordering inherited from X) has an upper bound in X. Then X has a maximal element.

In the proof of the theorem above one begins by choosing an upper bound for each well-ordered subset A of X. Since the collection of upper bounds for each A is non-empty (by hypothesis), it seems reasonable that this should be possible. Nevertheless, in any rigorous treatment, we would need to justify this possibility in the assumed (or previously deduced) properties of sets. There is in fact an axiom (called the Axiom of Choice) in most treatments of set theory that says exactly what we need.

**The Axiom of Choice**: Let C be any non-empty collection of non-empty sets. Then there exists a set that consists of exactly one member of each of the sets in C.

The Axiom of Choice is known to be logically independent of the other axioms of set theory, so it is reasonable to include it in a list of the fundamental axioms of the subject. It can be shown (see for example Paul Halmos' *Naïve Set Theory*) that Zorn's Lemma implies the Axiom of Choice. Thus the Theorem above, Zorn's Lemma, and the Axiom of Choice are all equivalent. There are in fact a great many other propositions (including the Tychonoff Product Theorem, the Hahn-Banach Theorem and the assertion that for every set S there exists a well-ordering of S) that turn out to be equivalent to the Axiom of Choice. Note that the Axiom of Choice is also equivalent to the assertion that the Cartesian product of any non-empty collection of non-empty sets is itself non-empty.