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Mathematics 1501 Hour Examination

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Directions: Do all problems. Show your work and justify your answers. Calculators are not allowed, and this is a closed book examination. You are allowed one prepared sheet of paper. Make sure your name is on all four pages of your examination.

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1 (30) Let $f(x) = x^4 - x^2$. Note that $f(x) = 0$ if and only if $x = -1$, $x = 1$ or $x = 0$.

a. Find and classify (as a local maximum, a local minimum, or neither) all of the critical points of f.

$f'(x) = 4x^3 - 2x$, so $f'(x) = 0 \iff$

$4x^3 - 2x = 0 \iff x = 0$ or $x^2 = \frac{1}{2} \iff x = 0, \pm\frac{1}{\sqrt{2}}$

$f''(x) = 12x^2 - 2$

$f''(0) = -2 < 0$, so 0 gives a local maximum

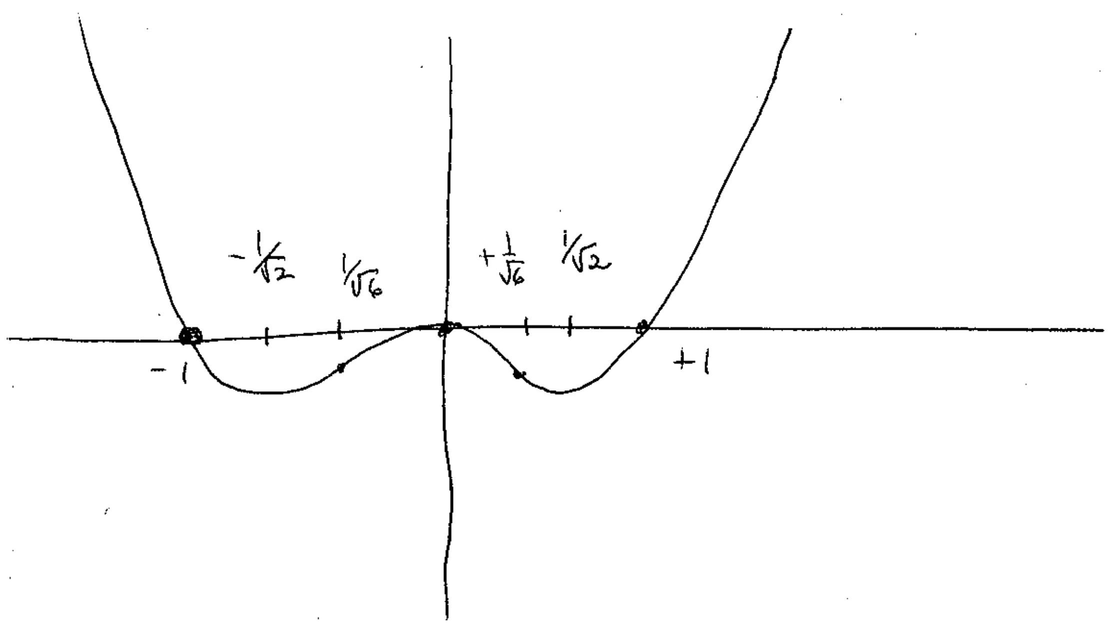
$f''\left(\pm\frac{1}{\sqrt{2}}\right) = 12\left(\frac{1}{2}\right) - 2 > 0$, so $\pm\frac{1}{\sqrt{2}}$ gives local minima

b. Find all the inflection points of f (i.e., all the points where the concavity of the graph of f changes).

$f''(x) = 12x^2 - 2 = 2(6x^2 - 1) \geq 0 \iff x^2 \geq \frac{1}{6} \iff |x| \geq \frac{1}{\sqrt{6}}$

Thus $f''(x)$ changes sign at $-\frac{1}{\sqrt{6}}$ and at $+\frac{1}{6}$, so these are the inflection points

c. Give a rough sketch of the graph of f, showing the local maxima and minima and the concavity properties of the graph.



2 (30) Let $f(x) = \dfrac{x^4+1}{x}$.

a. Find all of the asymptotes of f. $f(x) = x^3 + \frac{1}{x}$ has a vertical asymptote at $x = 0$. There are no other asymptotes.

b. Where exactly is the function f increasing? Where is it decreasing? Find and classify all of the critical points of f.

$f'(x) = 3x^2 - \frac{1}{x^2}$ so $f'(x) > 0 \iff 3x^4 > 1 \iff x^4 > \frac{1}{3}$

$\iff x^2 > \frac{1}{\sqrt{3}} \iff x > \frac{1}{\sqrt[4]{3}}$ or $x < -\frac{1}{\sqrt[4]{3}}$. Similarly

$f'(x) < 0 \iff -\frac{1}{\sqrt[4]{3}} < x < \frac{1}{\sqrt[4]{3}}$ and $x \neq 0$.

f increases on $\left(-\infty, -\frac{1}{\sqrt[4]{3}}\right)$ and on $\left(\frac{1}{\sqrt[4]{3}}, +\infty\right)$

f decreases on $\left(-\frac{1}{\sqrt[4]{3}}, 0\right)$ and on $\left(0, \frac{1}{\sqrt[4]{3}}\right)$

c. Where exactly is the function concave up and where concave down, and why?

$f''(x) = 6x + 2x^3 = x(6 + 2x^2)$

$f''(x) > 0$ if $x > 0$ and $f''(x) < 0$ if $x < 0$

Thus f is concave up on $(0, +\infty)$ and

f is concave down on $(-\infty, 0)$

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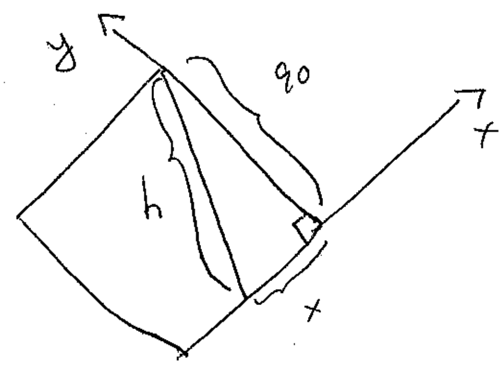
3. (20) A baseball diamond is a square with sides that are 90 feet long. Home plate, first base, second base, and third base occupy the vertices of this square, with home plate and second base opposite one another, and with first base and third base opposite one another. A batter hits the ball and runs from home plate toward first base with a speed of 24 ft/sec. At what rate is his distance from second base decreasing when he is half way between home plate and first base?



$$x^2 + 90^2 = h^2$$

and

$$\frac{dx}{dt} = 24$$

$$2x\frac{dx}{dt} + 0 = 2h\frac{dh}{dt}, \quad \text{so} \quad x\frac{dx}{dt} = h\frac{dh}{dt}$$

When the batter is half way between home and 1st base, $x = -45$, so that $h^2 = (-45)^2 + 90^2$

$$= \left(-\frac{90}{2}\right)^2 + 90^2 = 90^2\left(+\frac{1}{4}+1\right) = 90^2\left(\frac{5}{4}\right)$$

so that $h = 90\frac{\sqrt{5}}{2} = 45\sqrt{5}$. Then

$$(-45)(24) = (45\sqrt{5})\left(\frac{dh}{dt}\right), \quad \text{so}$$

$$-\frac{24}{\sqrt{5}} = \frac{dh}{dt}$$

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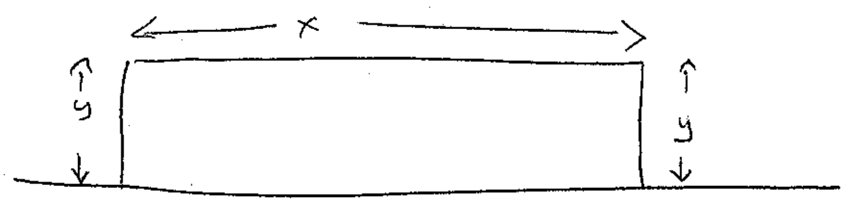
4 (20) A fence is to enclose a rectangle, one side of which lies along a wall. There is 1000 feet of fence available, and no fencing is required along the side that lies along the wall. What are the dimensions of the largest rectangle that can be enclosed in this way?



$1000 = x + 2y$, so $y = 500 - \frac{1}{2}x$

Area $= A(x) = xy = x\left(500 - \frac{1}{2}x\right)$, with $0 < x < 1000$

$A'(x) = \frac{d}{dx}\left(500x - \frac{1}{2}x^2\right) = 500 - x$

$A'(x) = 0 \iff 500 = x$

Since $A''(x) = -1$, $x = 500$ must give a maximum

When $x = 500$ ft, $y = 500 - \frac{1}{2}(500) = 250$ ft.