Mathematics 4317 Hour Examination – Sept. 26, 2007

Directions: Do any **four** of the following five problems. Show your work, and justify your answers and assertions. If you do all five problems, indicate on your paper which four you wish to have graded. This is a closed book examination, and calculators are allowed. Throughout this examination, the symbol "**R**" will denote the real number system, and $\|\ \|$ and $\cdot$ will denote the usual norm and inner product on $\mathbf{R^p}$.

1. (25) Let f be a function with domain A and range contained in B, and let g be a function with domain B and range contained in C.

a) Show that if f and g are one-to-one (i.e., injections), then so is the composition of g and f.

b) Show that if f is onto B and g is onto C, (i.e.,if f and g are surjections), then the composition of g and f is onto C.

c) Show that if there exist a bijection between A and B and a bijection between B and C, then there exists a bijection between A and C.

2. (25) Use the properties and definitions of the norm and inner product in $\mathbf{R^p}$ to prove the following:

a) For all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{R^p}$, $\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)$.

b) Show that $\|\mathbf{x} + \mathbf{y}\| = \|\mathbf{x} - \mathbf{y}\|$ if and only if $\mathbf{x}\cdot\mathbf{y} = 0$. Interpret this geometrically.

c) Show that for all $\mathbf{x} = (x_1, x_2, \ldots, x_p)$, we have $|x_1| + |x_2| + \ldots + |x_p| \leq p^{1/2}\,\|\mathbf{x}\|$. [Hint: Use the Cauchy-Schwarz Inequality.]

3. (25)

a) State carefully the Bolzano-Weierstrass Theorem.

b) Give a precise definition of "x is a cluster point of S".

c) Prove that a set is closed if and only if it contains all of its cluster points.

4. (25) The *interior* of a subset S of $\mathbf{R^p}$ is the union of all the open subsets of $\mathbf{R^p}$ that are contained in S.

a) The interior of S is always open. Why?

b) Show that if U is an open subset of $\mathbf{R^p}$ and U is contained in S, then U is contained in the interior of S.

c) Show that a subset of $\mathbf{R^p}$ is open if and only if it is equal to its interior.

d) In **R**, what is the interior of the set of all rational numbers? Why?

5. (25) The *symmetric difference* $A\Delta B$ of A and B is the set $(A\backslash B)\cup(B\backslash A)$.
Show that $(A\Delta B)\Delta C = A\Delta(B\Delta C)$. [Hint: Show that $(A\Delta B)\Delta C$ is the set consisting of all those x that lie in an odd number of the sets A, B, and C.]

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1a) Let $x, y \in A$. Suppose $g \circ f(x) = g \circ f(y)$. Then $g(f(x)) = g(f(y))$. Since g is one-to-one, $f(x) = f(y)$. Since f is one-to-one, $x = y$. Thus $g \circ f$ is one-to-one.

b) Let $z \in C$. Since $g: B \to C$ is onto, there exists $y \in B$ with $z = g(y)$. Since $f: A \to B$ is onto, there exists $x \in A$ with $y = f(x)$. Then $z = g(y) = g(f(x)) = g \circ f(x)$. Thus $g \circ f$ is onto

c) Suppose $f: A \to B$ and $g: B \to C$ are bijections. By parts a) and b) above, $g \circ f: A \to C$ is a bijection.

2 a) $\|x+y\|^2 + \|x-y\|^2 = (x+y)\cdot(x+y) + (x-y)\cdot(x-y) =$
$x\cdot x + 2x\cdot y + y\cdot y + x\cdot x - 2x\cdot y + y\cdot y = 2\,x\cdot x + 2\,y\cdot y = 2\|x\|^2 + 2\|y\|^2 =$
$2(\|x\|^2 + \|y\|^2)$

b) $\|x+y\| = \|x-y\| \iff \|x+y\|^2 = \|x-y\|^2 \iff (x+y)\cdot(x+y) = (x-y)\cdot(x-y)$
$\iff x\cdot x + 2x\cdot y + y\cdot y = x\cdot x - 2x\cdot y + y\cdot y \iff 2x\cdot y = -2x\cdot y \iff x\cdot y = 0$.
This says the diagonals of a parallelogram are of equal length if and only if the parallelogram is a rectangle.

c) $|x_1| + |x_2| + \dots + |x_p| = (|x_1|, |x_2|, \dots, |x_p|)\cdot(1, 1, \dots 1) \le$
$\|(|x_1|, \dots, |x_p|)\|\,\|(1, \dots, 1)\| = \|x\|\sqrt{p}$

3 See your text

4 a) Any union of open sets is open

b) If $U \subseteq S$ and U is open, then U is one of the sets over which we take the union to get the interior of S. Thus $U \subseteq$ interior of S

c) If S is open, then $S \subseteq S$ and S is open, so by part b), $S \subseteq$ interior of S. But the interior of S is a union of subsets of S, so is contained in S. Thus S = interior of S whenever S is open. Conversely, suppose S equals its interior. Since the interior is always open, S is open

5 [over to the next page.]

5. Observe first that if $x \in A \Delta B$, then $x \in A$ or $x \in B$ but not both. Thus $A \Delta B$ is the set of all x that lie in precisely one of A and B.

Now consider $x \in (A \Delta B) \Delta C$. Then $x \in (A \Delta B) \setminus C$ or $x \in C \setminus (A \Delta B)$.

Case I: Suppose $x \in (A \Delta B) \setminus C$. Then $x \notin C$ and $x \in A \Delta B$. Thus $x \notin C$ and x lies in exactly one of A and B. Thus x lies in exactly one of A, B and C.

Case II: Suppose $x \in C \setminus (A \Delta B)$. Then $x \in C$ and $x \notin A \Delta B$. Then $x \in C$ and either x lies in both of A and B or x lies in neither of A and B. If x lies in both of A and B, then x lies in all three of A, B and C. If x lies in neither of A and B, then x lies in exactly one of A, B and C.

Thus we see that if $x \in (A \Delta B) \Delta C$, then in every case x lies in an odd number of the sets A, B and C.

Conversely, suppose x lies in an odd number of the sets A, B, and C. Suppose first that $x \in A$ but $x \notin B$ and $x \notin C$. Then $x \in A \setminus B$ but $x \notin C$, so $x \in A \Delta B$ but $x \notin C$, so $x \in (A \Delta B) \setminus C \subseteq (A \Delta B) \Delta C$. Similarly, if $x \in B$ but $x \notin A$ and $x \notin C$, then $x \in (A \Delta B) \Delta C$. Suppose then that $x \in C$ but $x \notin A$ and $x \notin B$. Then $x \in C$ and $x \notin A \Delta B$, so $x \in C \setminus (A \Delta B) \subseteq (A \Delta B) \Delta C$. This shows that if x lies in exactly one of A, B and C, then $x \in (A \Delta B) \Delta C$. Suppose finally that x lies in all three of A, B and C. Then $x \in C$ but $x \notin A \Delta B$, so $x \in C \setminus (A \Delta B) \subseteq (A \Delta B) \Delta C$.

We have shown then that if x lies in an odd number of A, B, and C, then $x \in (A \Delta B) \Delta C$. It follows that $(A \Delta B) \Delta C$ is exactly the set of all those x that lie in an odd number of the sets A, B and C.

Now from the definition of $A \Delta B$, we see that $A \Delta B = B \Delta A$, so the symmetric difference is a commutative operation on sets. Thus $A \Delta (B \Delta C) = (B \Delta C) \Delta A = \{x : x \text{ lies in an odd number of the sets } B, C, \text{ and } A\} = (A \Delta B) \Delta C$.