Mathematics 4317 Hour Examination – Oct. 26, 2007

Directions: Do each of the following problems. Show your work, and justify your answers and assertions. This is a closed book examination, and calculators are allowed. Throughout this examination, the symbol "**R**" will denote the real number system, and || || will denote the usual norm on $\mathbf{R}^p$.

1. (25) Show that a subset S of $\mathbf{R}^p$ is closed if and only if it has the following property: whenever $\{x_n\}_{n=1}^{+\infty}$ is a sequence of elements of S and $\{x_n\}_{n=1}^{+\infty}$ converges to a point z of $\mathbf{R}^p$, then z lies in S.

2. (25) Suppose A and B are non-empty compact subsets of $\mathbf{R}^p$.
a. Show that if A and B are disjoint, then $\inf\{\|x-y\| : x \in A, y \in B\} > 0$.
b. Show that if A and B are disjoint then there exist disjoint open subsets U and V of $\mathbf{R}^p$ such that A is contained in U and B is contained in V.
c. Show that if A ∪ B is connected, then A and B are not disjoint.

3. Let $\{x_n\}_{n=1}^{+\infty}$ be a sequence in $\mathbf{R}^p$, and let $s_n = \frac{1}{n}(x_1 + x_2 + ... + x_n)$.
a. Show that if $\{x_n\}_{n=1}^{+\infty}$ is bounded, then $\{s_n\}_{n=1}^{+\infty}$ has a convergent subsequence.
b. Show that if $x_n = (-1)^n$ in **R**, then $\{s_n\}_{n=1}^{+\infty}$ converges to zero.
c. Show that if $\{x_n\}_{n=1}^{+\infty}$ converges to a point z of $\mathbf{R}^p$, then $\{s_n\}_{n=1}^{+\infty}$ also converges to z.

4.
a. Show that $1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!}$ is a Cauchy sequence.
b. Let $f_n(x) = x^n(1-x)^n$, where $0 \le x \le 1$. Show that f_n converges on [0,1] to the zero function. Is the convergence uniform on [0,1]? Why or why not?

1. Suppose S is closed. Let $\{x_n\}_{n=1}^{+\infty}$ be a sequence of elements of S such that $\{x_n\}_{n=1}^{+\infty}$ converges to z. If z is not an element of S, then z is an element of the open neighborhood S^c, where S^c is the complement of S. But then there exists N such that whenever $n \geq N$, then also $x_n \in S^c$. Since each x_n lies in S, this would be a contradiction. Suppose conversely that S has the stated property for sequences. To show that S is closed, it suffices to show that S^c is open. So let $z \in S^c$. We claim that there exists an n such that the open ball $\{w \in \mathbf{R}^p : \|w - z\| < \frac{1}{n}\}$ contains no point of S, which will finish the proof. If the claim is not true, then for every n there exists $x_n \in S$ such that $\|x_n - z\| < \frac{1}{n}$. But then $\{x_n\}_{n=1}^{+\infty}$ is a sequence in S that converges to z and z is not in S, contradicting the hypothesis.

2a. Suppose that $\inf\{\|x - y\| : x \in A, y \in B\} = 0$. Then for every n, there exist $x_n \in A$ and $y_n \in B$ such that $\|x_n - y_n\| < \frac{1}{n}$. Then the sequence $\{\|x_n - y_n\|\}_{n=1}^{+\infty}$ converges to zero. Moreover, $\{x_n\}_{n=1}^{+\infty}$ is a sequence in the bounded set A, and so has a convergent subsequence $\{x_{n_k}\}_{k=1}^{+\infty}$. Since A is closed, the limit x of this subsequence must be in A. Consider the subsequence $\{y_{n_k}\}_{k=1}^{+\infty}$ of $\{y_n\}_{n=1}^{+\infty}$. This subsequence lies in the closed bounded set B, and so has a convergent subsequence Y whose limit is an element of B. Let X be the subsequence of $\{x_{n_k}\}_{k=1}^{+\infty}$ that uses the same indices as Y. Then X must also converge to x. We claim now that x = y, which means that A and B are not disjoint, a contradiction. For ease of notation, let us write $\{w_n\}_{n=1}^{+\infty}$ for X and $\{z_n\}_{n=1}^{+\infty}$ for Y. To prove the claim, we note that for all n we have $\|x - y\| \leq \|x - w_n\| + \|w_n - z_n\| + \|z_n - y\|$. Since X converges to x, Y converges to y, and $\{\|x_n - y_n\|\}_{n=1}^{+\infty}$ converges to zero, we must have $\|x - y\| = 0$, so x = y.

b. For each x in $\mathbf{R}^p$, let B_x be the open ball centered at x of radius d, where $0 < 2d < \inf\{\|x - y\| : x \in A, y \in B\}$. Let U be the union of all the B_x such that x lies in A and let V be the union of all the B_x such that x lies in B. Then A is contained in U and B is contained in V. If w lies in both of U and V, then z lies in B_x for some x in A and in B_z for some z in B. We then get $\|x - z\| \leq \|x - w\| + \|w - z\| < d$, which contradicts the definition of d.

c. If A and B are disjoint, then the sets U and V from part b would form a disconnection of the union of A and B.

3a. If $\sup\{\|x_n\| : n \geq 1\} \leq M$, then for all n, we have

$$\left\|\frac{1}{n}(x_1 + x_2 + \ldots + x_n)\right\| \leq \frac{1}{n}(\|x_1\| + \|x_2\| + \ldots + \|x_n\|) \leq \frac{nM}{n} = M.$$

By the Bolzano-Weierstrass Theorem, $\{s_n\}_{n=1}^{+\infty}$ has a convergent subsequence.

b. For each n the numerator of s_n is either one or zero. Then s_n has norm at most 1/n for every n.

c. For each n we have

$$\left\|\frac{1}{n}(x_1+x_2+\ldots+x_n)-x\right\|=\left\|\frac{1}{n}[(x_1-x)+(x_2-x)+\ldots+(x_n-x)]\right\|\le\frac{1}{n}(\|x_1-x\|+\|x_2-x\|+\ldots+\|x_n-x\|)$$

Choose N so that n ≥ N implies that $\|x_n-x\|<\frac{\varepsilon}{2}$. Then for n ≥ N, we have

$$\left\|\frac{1}{n}(x_1+x_2+\ldots+x_n)-x\right\|\le\frac{1}{n}(\|x_1-x\|+\|x_2-x\|+\ldots+\|x_N-x\|+(n-N)\frac{\varepsilon}{2})<\frac{1}{n}\{\|x_1-x\|+\|x_2-x\|+\ldots+\|x_N-x\|\}+\frac{\varepsilon}{2})$$

Since $\|x_1-x\|+\|x_2-x\|+\ldots+\|x_N-x\|$ is fixed, $\frac{1}{n}\{\|x_1-x\|+\|x_2-x\|+\ldots+\|x_N-x\|\}<\frac{\varepsilon}{2}$ for all sufficiently large n. It follows that for all sufficiently large n we have $\|s_n-x\|<\varepsilon$.

4a. Let n < m. Then

$$(1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{m!})-(1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!})=\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\ldots+\frac{1}{m!}<$$

$$\frac{1}{2^n}+\frac{1}{2^{n+1}}+\ldots+\frac{1}{2^{m-1}}=\frac{1}{2^n}(1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{m!})<\frac{1}{2^n}2=\frac{1}{2^{n-1}}<\frac{1}{n}$$

We choose N so that $\frac{1}{N}<\varepsilon$. Then for N < n < m we have

$$\left|(1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{m!})-(1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!})\right|<\varepsilon$$

b. For any x in [0,1], $x(1-x)$ is at most ½, since one of the two numbers x and 1 – x is at most ½ and the other is at most 1. Thus $f_n(x)=x^n(1-x)^n$ is at most $\left(\frac{1}{2}\right)^n$. Thus $\sup\{|f_n(x)|: x\in[0,1]\}\le\left(\frac{1}{2}\right)^n$. Since $\left(\frac{1}{2}\right)^n$ converges to zero, f_n converges uniformly to zero on [0,1].