NAME: ______________________________ Rec. leader: ____________________

Mathematics 2401 Hour Examination

W. L. Green

March 7, 2006

Directions: Do all problems. Show your work and justify your answers. This is a closed book examination, and calculators are allowed. Make sure your name is on all four pages of your examination.

Page	
1	
2	
3	
4	
Total	

1. (48) A circular plate occupies the space in the x-y-plane described by $x^2 + y^2 \leq 25$. Suppose the temperature at any point in the plate is given by the function $f(x,y) = xy$.

a. Find the gradient of f.

$$\nabla f(x,y) = (y, x)$$

b. Find the directional derivative of f at (2,3) in the direction of the unit vector u that points from (0,0) toward (1,-1).

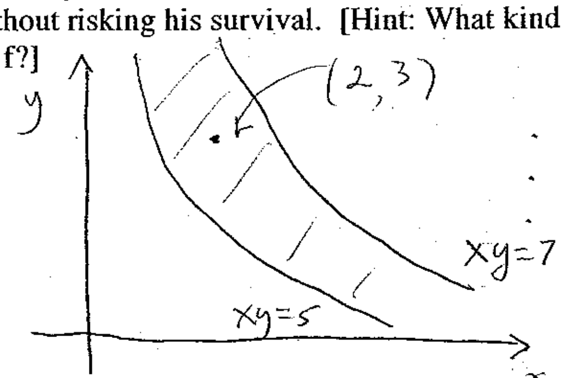
$$u = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \quad \text{and} \quad \nabla f(2,3) = (3,2)$$

$$(3,2)\cdot\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

c. In what direction is the temperature increasing most rapidly at the point (2,3)?

In the direction of the gradient (3, 2)

d. A certain bug can only survive if the temperature under his feet is between 5 degrees and 7 degrees. Suppose we put him down on the plate at the point (2,3). Sketch the region on the plate that the bug can reach without risking his survival. [Hint: What kind of curves are the level curves of the function f?]



1 (continued) A circular plate occupies the space in the x-y-plane described by $x^2 + y^2 \le 25$. Suppose the temperature at any point in the plate is given by the function $f(x,y) = xy$.

e. Another bug, more tolerant of temperature, runs in a straight line from the center of the plate to a point on the boundary. If he runs at a constant velocity of $(\frac{dx}{dt}, \frac{dy}{dt}) = (3,4)$, how fast is the temperature changing under his feet at time t?

$$\frac{d}{dt}(f(x,y)) = \nabla f(x,y) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (y, x) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$$

$(x,y) = (3t, 4t)$, so the rate of change is $(4t, 3t) \cdot (3, 4) = 24t$ or

$(x,y) = (3t, 4t)$, so $f(x,y) = (3t)(4t) = 12t^2$, which has derivative $24t$

f. Show that the hottest points on the plate must be on the boundary of the plate. [Hint: You don't have to find these points to show that they must lie on the boundary.]

At an interior point of the plate, we have $\nabla f(x,y) = (0,0)$ whenever (x,y) is a local minimum or maximum. But $(y,x) = (0,0)$ only at the origin, where the 2nd derivative is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. This is indefinite, so $(0,0)$ gives a saddle point.

2. (36) Let $f(x,y) = x^4 + y^4 - 4xy + 1$.

a. Find equations for the tangent line and the normal line to the curve f(x,y) = 7 at the point(1,-1).

$$\nabla f(x,y) = (4x^3 - 4y, 4y^3 - 4x) \text{ and } \nabla f(1,-1) = (8, -8)$$

The ~~normal~~ tangent line has equation $(8, -8) \cdot (x-1, y+1) = 0$, i.e. $y = x-2$

The ~~tangent~~ normal line has equation $(8, 8) \cdot (x-1, y+1) = 0$, i.e. $y = -x$

2. (continued) Let $f(x,y) = x^4 + y^4 - 4xy + 1$.
b. Find the local maximum and minimum values and saddle points of the function f.

$\nabla f(x,y) = (4x^3 - 4y,\ 4y^3 - 4x)$

$\nabla f(x,y) = (0,0) \iff x^3 = y$ and $y^3 = x$

These equations imply that $x = y^3 = (x^3)^3 = x^9$, i.e. that $x^9 - x = 0$, i.e. that $x(x^8-1)=0$. This holds if and only if $x=0$, $x=1$ or $x=-1$

The critical points are $(0,0)$ $(1,1)$ and $(-1,-1)$

The 2nd derivative matrix is $\begin{bmatrix} 8x^2 & -4 \\ -4 & 8y^2 \end{bmatrix}$

At $(0,0)$, this matrix is $\begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}$, which is indefinite

$(0,0)$ is a saddle point.

At $(1,1)$ the matrix is $\begin{bmatrix} 8 & -4 \\ -4 & 8 \end{bmatrix}$, which is positive definite, since $8>0$ and $\det\begin{bmatrix} 8 & -4 \\ -4 & 8 \end{bmatrix} > 0$

$-1 = f(1,1)$ is a local minimum value

At $(-1,-1)$, the matrix is the same as at $(1,1)$, so

$-1 = f(-1,-1)$ is a local minimum value.

There are no local maximum values

4 (20) Use the method of Lagrange multipliers to find the minimum and maximum values of the function $x^2 + y^2$ subject to the constraint $x^4 + y^4 = 1$.

$$(2x, 2y) = \lambda(4x^3, 4y^3) \iff$$

$$\left\{\begin{array}{l} 2x = 4\lambda x^3 \\ 2y = 4\lambda y^3 \end{array}\right\} \iff \left\{\begin{array}{l} x = 2\lambda x^3 \\ y = 2\lambda y^3 \end{array}\right\}$$

Either $x=0$ or $x^2 = \frac{1}{2\lambda}$ ($\lambda = 0$ means $x = y = 0$, so $x^4 + y^4 = 0 \neq 1$)

Either $y=0$ or $y^2 = \frac{1}{2\lambda}$

If $x \neq 0$ and $y \neq 0$, then $x^2 = y^2$, so $x^4 + y^4 = 1$ gives $2x^4 = 1$, i.e. $x^2 = \frac{1}{\sqrt{2}}$. Then $y^2 = x^2 = \frac{1}{\sqrt{2}}$

This gives a value of $2\frac{1}{\sqrt{2}} = \sqrt{2}$ for x^2+y^2

If $x=0$, then $y^4 = 1$, so $y^2 = 1$, so $x^2+y^2 = 1$

If $y=0$, then $x^4 = 1$, so $x^2 = 1$, so $x^2+y^2 = 1$

The minimum value is 1, and the maximum value is $\sqrt{2}$