Newton's Method in Several Variables
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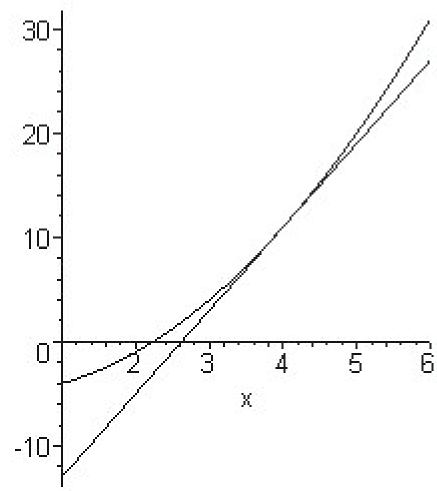
Newton's Method is a technique for approximating the solution to an equation or system of equations. You studied the one variable case in first semester calculus. It is based on the sketch below.

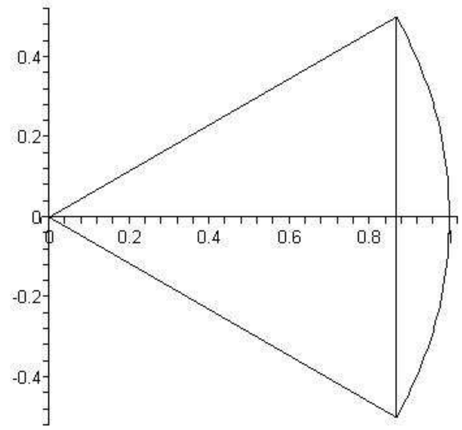


If we wish to find where the function *f* assumes the value zero, and we do so by making a sequence of approximations. As an initial approximation, we make a reasonable guess and call it x_0. In the example above, x_0 is 4. We then construct the tangent line at $(x_0, f(x_0))$ and calculate the *x*-intercept of this tangent line. This is our second approximation. We name it x_1, and use the same tangent line method to find the next approximation x_2. In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Newton's method does not always work, but when it does work it works extremely well, doubling the number of correct digits at each step.

We illustrate the multivariable situation with an example in two variables. It will look exactly like the one variable iteration, except that the *x*'s are vectors, the derivative is a matrix, and the matrix inverse replaces the reciprocal of the derivative. We discuss the method here and illustrate the calculations in a Maple worksheet. Suppose we have a lens, illustrated here in cross section



that we have measured the arc length to be 1.0000000, the chord length to be .9983341665, and we seek the radius *r* and semi-apex angle *t*. That is, we want to approximate the solution to the system of equations

$$\begin{aligned} 2\,r\,t - 1 &= 0 \\ 2\,r\sin(t) - .9983341665 &= 0 \end{aligned}.$$

We can rephrase this as the problem of solving

$$\mathbf{f}(r,t) = \begin{pmatrix} f_1(r,t) \\ f_2(r,t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $f_1(r,t) = 2\,r\,t - 1$ and $f_2(r,t) = 2\,r\sin(t) - .9983341665$. The derivative matrix is

$$\mathbf{f}\,' = \begin{pmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial t} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial t} \end{pmatrix}$$

and the iteration is

$$\begin{pmatrix} r_{n+1} \\ t_{n+1} \end{pmatrix} = \begin{pmatrix} r_n \\ t_n \end{pmatrix} - \left(\mathbf{f}\,'(r_n, t_n)\right)^{-1} \mathbf{f}(r_n, t_n)$$

The calculations are illustrated in a Maple worksheet.