Mathematics 4318 Hour Examination –Feb. 22, 2008

Directions: Each part of each problem counts 12 points. Do any **eight** parts. (You will get the remaining 4 points for your name, so don't forget to put it on your paper.) Show your work, and justify your answers and assertions. This is a closed book examination, and calculators are allowed. Throughout this examination, "**R**" will denote the real number system.

1. For each of the following series, decide whether, or where, the series converges. Does it converge absolutely? Is the convergence in part b) uniform on the interval of convergence?

a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+2}}$

b) $\sum_{n=1}^{\infty} \frac{x^{n-1}}{2^n n^3}$

c) $\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^n} \frac{1}{n^2}$.

2. If f is the sum of the series in problem 1b, show that $\int_{-2}^{2} f(x)dx = 2 \sum_{\text{n is odd}} \frac{1}{n^4}$.

3. Suppose f is differentiable everywhere and that f is periodic in 2π.

a) Show that the derivative of f is also periodic in 2π.

b) Show that if the derivative of f is continuous everywhere, then the Fourier coefficients $a_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(t)\cos nt\,dt$ for the derivative and the Fourier coefficients $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$ for the function f satisfy $a_n' = nb_n$.

4.

a) Prove that if two functions have the same derivative on an interval I, then they differ by a constant on I.

b) Give an example to show that if D is not an interval, then it is possible for two functions to have the same derivative on D, yet not differ by a constant on D.

5.Let g be defined by g(0) = 1 and g(x) = 0 for x not equal to zero.

a) Show that if f is continuous on [-1,1], then f is integrable with respect to g over [-1,1].

b) Calculate $\int_{-1}^{1} f dg$ and $\int_{-1}^{1} g df$.

1a) $\frac{1}{\sqrt{n+2}} \downarrow 0$, so the series converges by the alternating series test. Since $\frac{1}{\sqrt{n+2}} = \frac{1}{(n+2)^{1/2}}$ and $\frac{1}{2} < 1$, the series $\sum \frac{1}{\sqrt{n+2}}$ diverges.

b) $\left| \frac{x^{n-1}}{2^n n^3} \right|^{1/n} = |x|^{1-\frac{1}{n}} \frac{2}{((n)^{1/n})^3} \to 2|x|$, so the radius of convergence is 2

At $x = \pm 2$, we have $\sum \frac{(\pm 2)^{n-1}}{2^n n^3} = \sum \frac{(\pm 1)^{n-1}}{2n^3}$. Now $\sum \frac{1}{n^3}$ is convergent,

so $\sum \frac{x^{n-1}}{2^n n^3}$ is absolutely convergent at $x = \pm 2$. Since $\left| \frac{(\pm 1)^{n-1}}{2n^3} \right| \leq \frac{1}{2n^3}$, it converges uniformly on $[-2, 2]$ by the Weierstrass M-test.

c) $\frac{(n+1)^n}{n^n} \frac{1}{n^2} = (1+\frac{1}{n})^n \frac{1}{n^2}$. Now $(1+\frac{1}{n})^n \uparrow e$ and $\sum \frac{1}{n^2}$ converges, so this series converges by Abel's test: The convergence is clearly absolute.

2 $\int_{-2}^{2} \sum_{n=1}^{\infty} \frac{x^{n-1}}{2^n n^3} dx = \sum_{n=1}^{\infty} \int_{-2}^{2} \frac{x^{n-1}}{2^n n^3} dx = \sum_{n=1}^{\infty} \left[\frac{x^n}{2^n n^4} \right]_{-2}^{2} = \sum_{n=1}^{\infty} \frac{1}{n^4} - \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^4}$

$= \left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots\right) - \left(\frac{-1}{1^4} + \frac{+1}{2^4} + \frac{-1}{3^4} + \dots\right) = 2\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\right)$

The interchange of integral and sum is justified by the fact that the series converges uniformly on $[-2, 2]$.

3 a) Let $c \in \mathbb{R}$. Then for $x \in \mathbb{R}$, $\frac{f(x+2\pi) - f(c+2\pi)}{(x+2\pi) - (c+2\pi)} = \frac{f(x) - f(c)}{x - c}$, by periodicity.

The first quotient tends to $f'(c+2\pi)$, the second to $f'(c)$, as $x \to c$.

b) See your text for this (p. 332).

4 a) Let $f' = g'$ on I. Then $(f-g)' = 0$ on I. Let $h = f - g$. For x_1 and x_2 in I, we have $h(x_1) - h(x_2) = (h'(c))(x_1 - x_2) = (0)(x_1 - x_2)$ for some $c \in I$.

Thus $h(x_1) = h(x_2)$, so h is constant on I. Thus $f - g$ is constant on I

b) Let $f(x) = 0$ if $x \leq 0$ and $f(x) = 1$ for $x > 0$. Let $g = 0$ on $\mathbb{R}$. Then $f' = g'$ on $(-1,0) \cup (1,2)$.

5 a) Let $\varepsilon > 0$. Choose $\delta > 0$ such that $|x - 0| < \delta$ implies $|f(x) - f(0)| < \varepsilon/2$. Now choose a Partition P_ε of $[-1, 1]$ such that $0 \in P_\varepsilon$ and $\max_k x_k - x_{k-1} < \delta$. Then the same is true for any partition P that refines P_ε. For any Riemann-Stieltjes sum $S = S(P, f, g)$, we have $S = f(\xi)(1-0) + f(\eta)(0-1) = f(\xi) - f(\eta)$ with $|\xi - 0| < \delta$ and $|\eta - 0| < \delta$.

Thus $|f(\xi) - f(0)| < \frac{\varepsilon}{2}$ and $|f(\eta) - f(0)| < \varepsilon/2$. Thus $|f(\xi) - f(\eta)| \leq |f(\xi) - f(0)| + |f(0) - f(\eta)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

It follows that $|S| = |S - 0| < \varepsilon$, and hence that $\int_{-1}^{1} f\,dg = 0$

b) As above $\int_{-1}^{1} f\,dg = 0$. Then $\int_{-1}^{1} g\,df = f(1)g(1) - f(-1)g(-1) - 0 = 0 - 0 - 0 = 0$