Math 4317 Midterm Solutions – Summer 2006

1. The rational numbers $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots, \frac{1}{3^n}, \ldots$ are all in The Cantor set, so the Cantor set contains infinitely many rationals. Since the rationals are countable, the Cantor set contains countably many rationals. If the set of irrational members of the Cantor set were countable, then the Cantor set would be countable, which is not true. Thus the Cantor set contains ~~infinitely~~ uncountably many irrationals.

2a) $\|x+y\|^2 + \|x-y\|^2 = (x+y)\cdot(x+y) + (x-y)\cdot(x-y)$
$= x\cdot x + 2x\cdot y + y\cdot y + x\cdot x - 2x\cdot y + y\cdot y = 2x\cdot x + 2y\cdot y =$
$2\|x\|^2 + 2\|y\|^2$

b) $(1,0)$ is non-zero, yet $(1,0) * (1,0) = (1)(0) = 0$

c) $|x_1| + \cdots + |x_p| = |x_1|(1) + \cdots + |x_p|(1) = (|x_1|, \ldots, |x_p|)\cdot(1,1,\ldots,1)$
$\le \left(\sqrt{|x_1|^2 + \cdots + |x_p|^2}\right)\left(1^2 + \cdots + 1^2\right)^{\frac{1}{2}} = \|\underline{x}\|\sqrt{p}$ by Cauchy–Schwarz

3 a) See text

b) Let $S = \{0\}$. Then $\sup S = 0$, but S has no cluster points

c) See text

d) We show first by induction that $n \le 2^n$ for all n.
If $n=1$, this says $1 \le 2$, which is true. Suppose then that $k \le 2^k$. Then $k+1 \le k+k = 2k \le 2(2^k) = 2^{k+1}$. Thus $n \le 2^n$ for all $n \ge 1$.
It follows then that $0 \le \frac{1}{2^n} \le \frac{1}{n}$ for all n
By the Archimedean property, there exists n with $\frac{1}{z} < n$, so that $\frac{1}{2^n} \le \frac{1}{n} < z$.

4 a) See text

b) $K \setminus U = K \cap U^c$. Since U is open, U^c is closed. Thus $K \cap U^c$ is closed. Now K is compact, so K is bounded. But $K \cap U^c \subseteq K$, so $K \cap U^c$ is also bounded. Thus $K \setminus U = K \cap U^c$ is closed and bounded. By the Heine–Borel theorem, $K \setminus U$ is compact.

5 Suppose $W \cap V$ is not empty, and let $z \in W \cap V$. Then for any $x \in W$ there exists a polygonal path in W from x to z, and for any $y \in V$ there is a polygonal path in V from z to y, since W and V are open and connected. It follows that there is a polygonal path in $W \cup V$ from x to y. Clearly any two elements of W or of V can be connected by a polygonal path in W or in V. Thus for any two points in $W \cup V$, there is a polygonal path in $W \cup V$ which connects them. Thus $W \cup V$ is connected.
Conversely, suppose $W \cap V$ is empty. Then since W and V ~~are~~ are open non-void and have union equal to $W \cup V$, they form a disconnection of $W \cup V$.