Mathematics 6580 Final Examination – July 31, 2007

Directions: Do all problems. Show your work, and justify your answers and assertions. This is a closed book examination, and calculators are allowed. Throughout this examination, the symbol "**C**" will denote the complex number system, $\| \ \|$ and < , > will denote norms and inner products respectively, and "show" means "prove." There are two pages and a total of 200 points in this examination.

1. (20) Let P be a pre-Hilbert space. State and prove **either** the Parallelogram Law **or** the Polarization Identity for P.

2. (20) Let $\{x_n\}$ be an orthonormal sequence in a Hilbert space H, and let $\{\lambda_n\}$ be a sequence of complex numbers.

a) Show that for any $m \geq 1$ we have $\left\|\sum_{n=1}^{m} \lambda_n x_n\right\|^2 = \sum_{n=1}^{m} |\lambda_n|^2$.

b) Show that the series $\sum_{n=1}^{+\infty} \lambda_n x_n$ converges to an element of H if and only if $\sum_{n=1}^{+\infty} |\lambda_n|^2 < +\infty$.

3. (20) Let H be the Hilbert space $\mathbf{C}^n$, and let K = $\{x = (x_1, \ldots, x_n) \in H : \sum_{i=1}^{n} x_i = 1\}$. What is the point of K that is nearest to zero and why?

4. (20) Let H be a Hilbert space with inner product and norm denoted by < , > and $\| \ \|$, and let K be the Hilbert space H⊕H, i.e., the space of all ordered pairs (x,y) where x and y lie in H, with the inner product on K given by

$$[(x,y),(u,v)] = \langle x,u\rangle + \langle y,v\rangle.$$

a) Show that for all (x,y) in K the norm $|||(x,y)|||$ of (x,y) satisfies $|||(x,y)|||^2 = \|x\|^2 + \|y\|^2$.

b) Show that if T is a bounded linear function from H into H, then the set {(x,T(x)) : x ∈ H} (this is the graph of T) is a closed linear subspace of K.

5. (20) Let ϕ be the sesquilinear form on C[0,1] defined by $\phi(f,g) = f(0)\overline{g(0)} + f(1)\overline{g(1)}$. Show that ϕ satisfies $|\phi(f,g)|^2 \leq \phi(f,f)\phi(g,g)$ for all f and g in C[0,1].

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6. (100) Let S and T be operators on a Hilbert space H. For each of the following give either a proof or a counterexample.

a) If S and T are isometric, then so is ST.

b) If S is isometric, then so is S*.

c) If S and T are unitary, then so is ST.

d) If S is compact (completely continuous), then S*S and SS* are compact.

e) If T is self-adjoint and N is a closed T-invariant subspace of H, then N reduces T.

f) If S and T are self-adjoint operators and ST = 0, then S and T have orthogonal ranges.

g) If S and T are non-zero projection operators and ST = 0, then $\|S+T\| < \|S\| + \|T\|$.

h) If S is normal, then S and S* have the same null-space.

i) If S is self-adjoint and λ is an eigenvalue for S, then λ is real.

j) If S is isometric and λ is an approximate eigenvalue for S, then $|\lambda| \leq 1$.