MATH 2401 - Harrell

A very gradifying lecture

Lecture 11

Scenes from our previous episode

- Tangent and normal vectors,
- Tangent and normal lines

$$z = f(x, y)$$

$$F(x, y, z) := z - f(x, y) = 0$$

N to graph $\quad -\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + 1\hat{k}$

Tips for the test – don't lose points for trivial reasons!

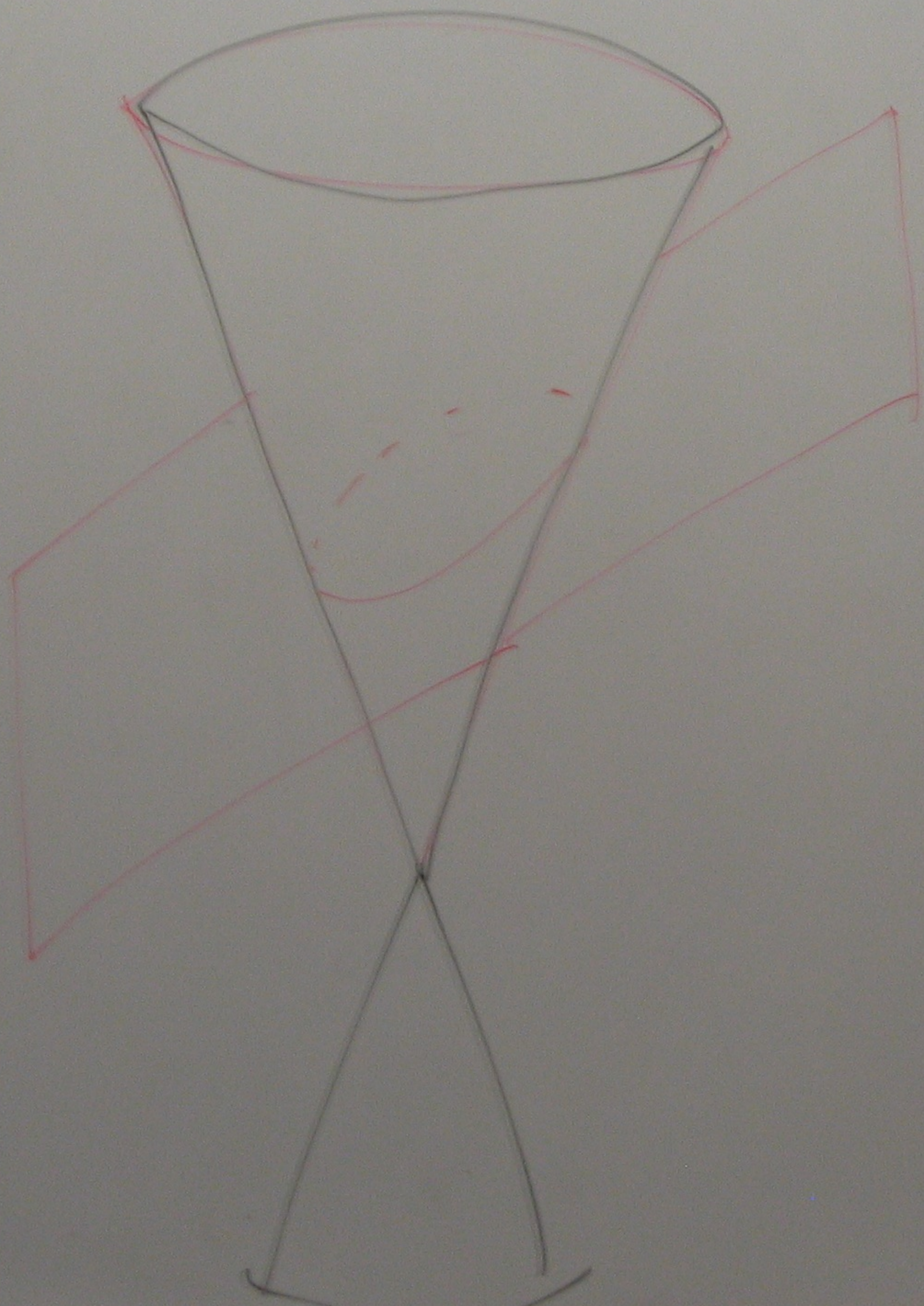
- Show some work – There will be no partial credit for what isn't shown.
  - Make sure the grader sees the key facts /formulae somewhere
- Check your algebra/calculations.
- Common sense. An answer has to be the "right kind of animal."
- Put the answer where indicated.

Monday was fun, so let's do some new (even more fiendishly intricate) geometric problems

- Like.....intersecting surfaces.
- The intersection of two surfaces is usually a curve. How is it oriented – i.e., what is its tangent direction?

Intersection of a plane and a cone

- Tangent to the curve at the intersection of 2 surfaces, such as a plane and a cone – the classic *conic sections*.



$$x^2 + y^2 = z^2$$

Example. $\{z^2 = x^2+y^2\} \cap \{x=1\}$

Method #1. Work out curve and differentiate.

$$\vec{r}(t) = (1, t, \sqrt{1+t^2})$$

$$\vec{r}\,'(t) = \left(0, 1, \frac{t}{\sqrt{1+t^2}}\right) \text{ (or any multiple, exc. } \times 0\text{)}$$

The tangent line at $(1, y, \sqrt{1+y^2})$ is

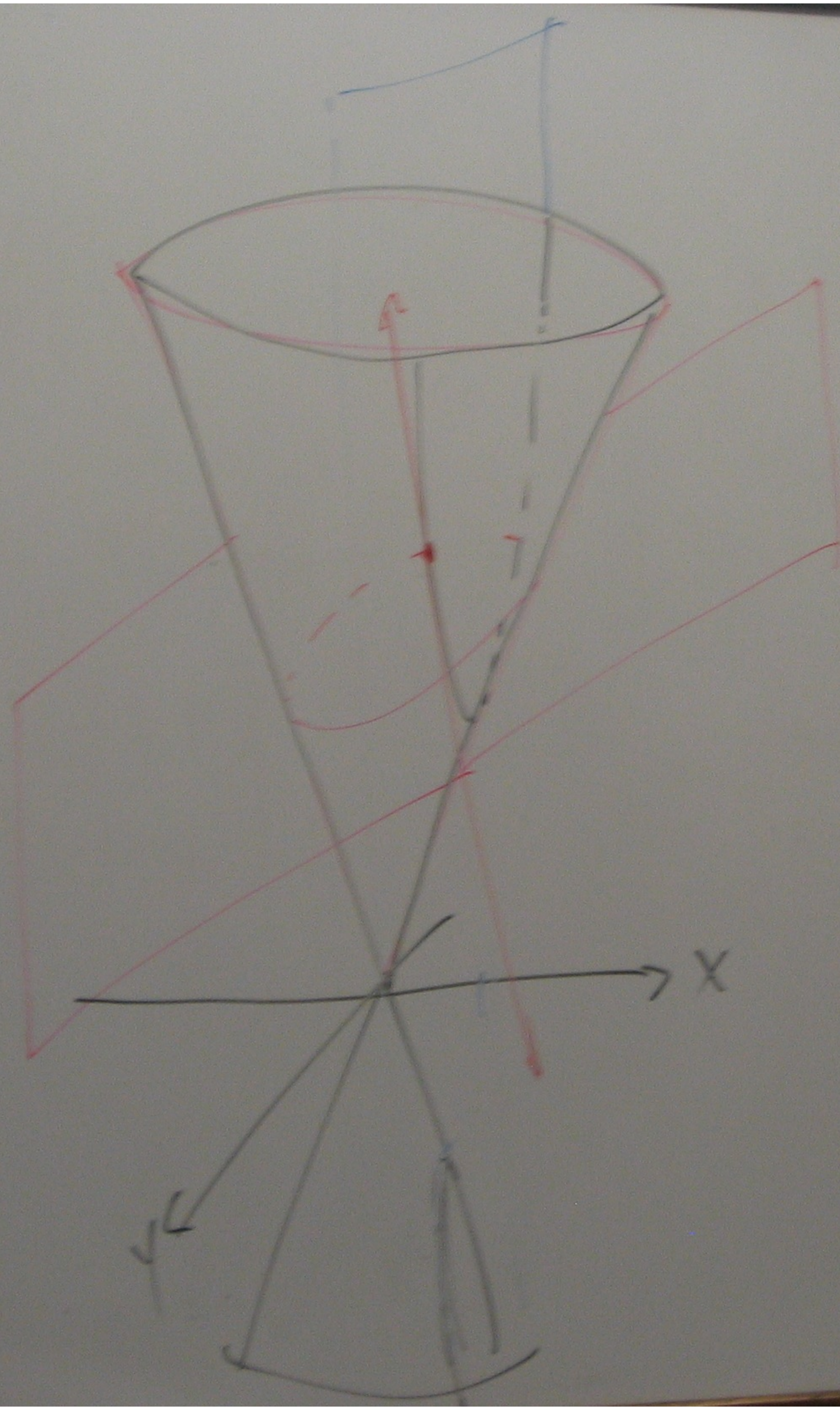
$$\{ (x,y,z) = (1, y, \sqrt{1+y^2}) + u\,(0, \sqrt{1+y^2}, y)$$

$$= (1, y + u\sqrt{1+y^2}, \sqrt{1+y^2} + uy)$$

Method #2. Find 2 normal vectors to surfaces and take cross product:

$$N_1 = \left(-\frac{x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1\right), \quad \hat{N}_2 = \hat{i}, \quad x \to 1$$

$$\vec{N}_1 \times \vec{N}_2 = \left(0, 1, \frac{y}{\sqrt{1+y^2}}\right)$$



$$x^2 + y^2 = z^2$$

But what about…

- The tangent to the curve where

  $x = \sin(\pi\, y\, z)$

  and

  $z = 2\, x^2 - 4\, y^2 + \frac{1}{4}$

  as it passes through $(1, \frac{1}{4}, 2)$?

$$x - \sin(\pi y z) = 0$$

$$N_1 = 1\,\hat{i} + (-\pi z \cos(\pi y z))\,\hat{j} + (-\pi y \cos(\pi y z))\,\hat{k}$$

$$\hat{N}_1(P) = \hat{i}$$

$$\hat{N}_2 = -4x\,\hat{i} + 8y\,\hat{j} - 1\,\hat{k}$$

$\hat{i} \times \hat{j} = \hat{k}$
$\hat{j} \times \hat{k} = \hat{i}$
$\hat{k} \times \hat{i} = \hat{j}$

@ pt.

$$N_2 = -4\hat{i} + 2\hat{j} + \hat{k}$$

A Tan to curve
$\equiv$

$$N_1 \times N_2 = -\hat{j} + 2\hat{k}$$

The chain rule(s)

- $(d/dt)\, f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$
- *Just like 1-D*
- In components:

  $df/dt = (\partial f/\partial x)\, dx/dt + (\partial f/\partial y)\, dy/dt$

  $+ (\partial f/\partial z)\, dz/dt + \ldots$

- Examples?

The chain rule(s)

- Suppose the temperature in a plate is $T(x,y) = 4x^2 - 2xy - 4y^2$, and that an object moves in a circle,

  $\mathbf{r}(t) = 2\cos(t)\,\mathbf{i} + 2\sin(t)\,\mathbf{j}$.

At what rate is the temperature changing?

$$\frac{dT}{dt} = \begin{bmatrix} 16\cos(t) - 4\sin(t) \\ -4\cos(t) - 16\sin(t) \end{bmatrix} \cdot \begin{pmatrix} -2\sin(t) \\ 2\cos(t) \end{pmatrix}$$

$$\nabla T(x, y) = (8x - 2y)\,\hat{i} + (-2x - 8y)\,\hat{j}$$

$$\vec{r}(t) = 2\cos(t)\,\hat{i} + 2\sin(t)\,\hat{j}$$

$$\vec{r}\,'(t) = -2\sin(t)\,\hat{i} + 2\cos(t)\,\hat{j}$$

$$\frac{dT}{dt} = \begin{bmatrix} 16\cos(t) - 4\sin(t) \\ -4\cos(t) - 16\sin(t) \end{bmatrix} \cdot \begin{pmatrix} -2\sin(t) \\ 2\cos(t) \end{pmatrix}$$

$$= 8\left(\sin^2(t) - 4\sin(t)\cos(t) - \cos^2(t) - 4\sin(t)\cos(t) \right)$$

$$= 8\left(\sin^2(t) - \cos^2(t) - 8\sin(t)\cos(t) \right)$$

The chain rule(s)

- What about $u(x,y)$, where x and y depend on s and t?
- For example, a change of variables.

The chain rule(s)

- What about u(x,y), where x and y depend on s and t?

$$(\partial u/\partial s) = (\partial u/\partial x)(\partial x/\partial s) + (\partial u/\partial y)(\partial y/\partial s)$$

- *Remember: Add up all the possible routes for connecting u to the independent variable.*

Word problems

- Suppose that price of your widgets is $P(t)$, you are selling at a rate of $R(t)$ per month, and your expenses are

  $F(t) + c(t)\,R(t)$

- How rapidly is your profit changing, if $P = 2$, $R = 3000$, $F = 2500$, $c = 1$, $P'(t) = .1$, $R'(t) = -20$, $c(t) = .05$, and $F'(t) = 5$?

Word problems

- Suppose that price of your widgets is $P(t)$, you are selling at a rate of $R(t)$ per month, and your expenses are

  $F(t) + c(t)\,R(t)$

- Profit = PR - F - c R

  $d\,\text{Profit}/dt = R\,P' + (P - c)\,R' - F' - c'R$

Mean value theorem

- If f is differentiable on the line segment **ab**, there exists a position **c** so that

  $$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \bullet (\mathbf{b} - \mathbf{a})$$

- Why is this not as great as in 1D?

  A: the gradient doesn't have to point in the direction $(\mathbf{b} - \mathbf{a})$, and $|\nabla f(\mathbf{c})|$ isn't the average slope $|f(\mathbf{b}) - f(\mathbf{a})| / |\mathbf{b} - \mathbf{a}|$.

Mean value theorem

- If f is differentiable on the line segment **ab**, there exists a position **c** so that

  $$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \bullet (\mathbf{b} - \mathbf{a})$$

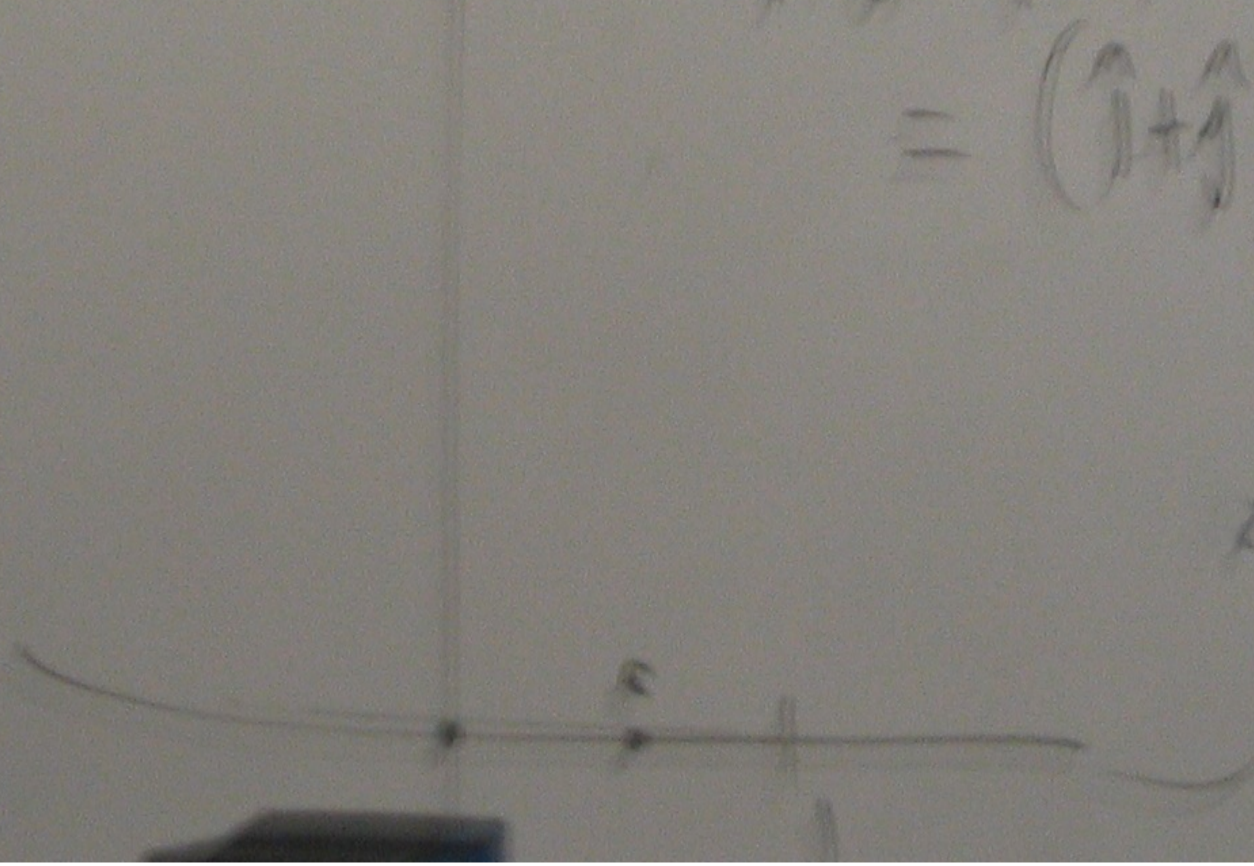
- Example: (0,0) to (1,0), f(x,y) = x+y.

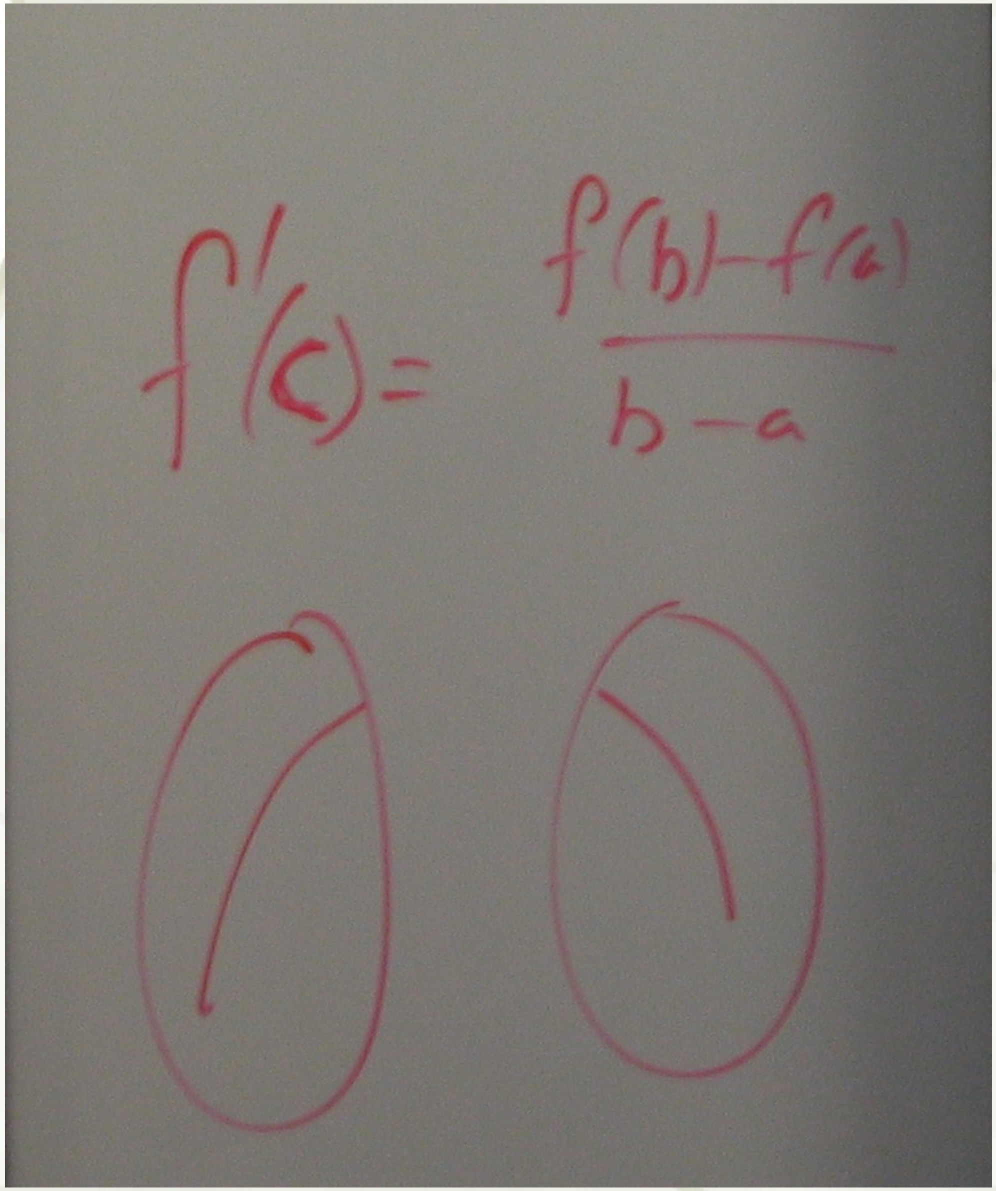
$\exists$ some place $\vec{c}$

between $\vec{0}$ and $(1, 0)$

s.t. $f(b) - f(a) = 1 - 0 = 1$

$= (\hat{\imath} + \hat{\jmath}) \cdot \hat{\imath} = 1$



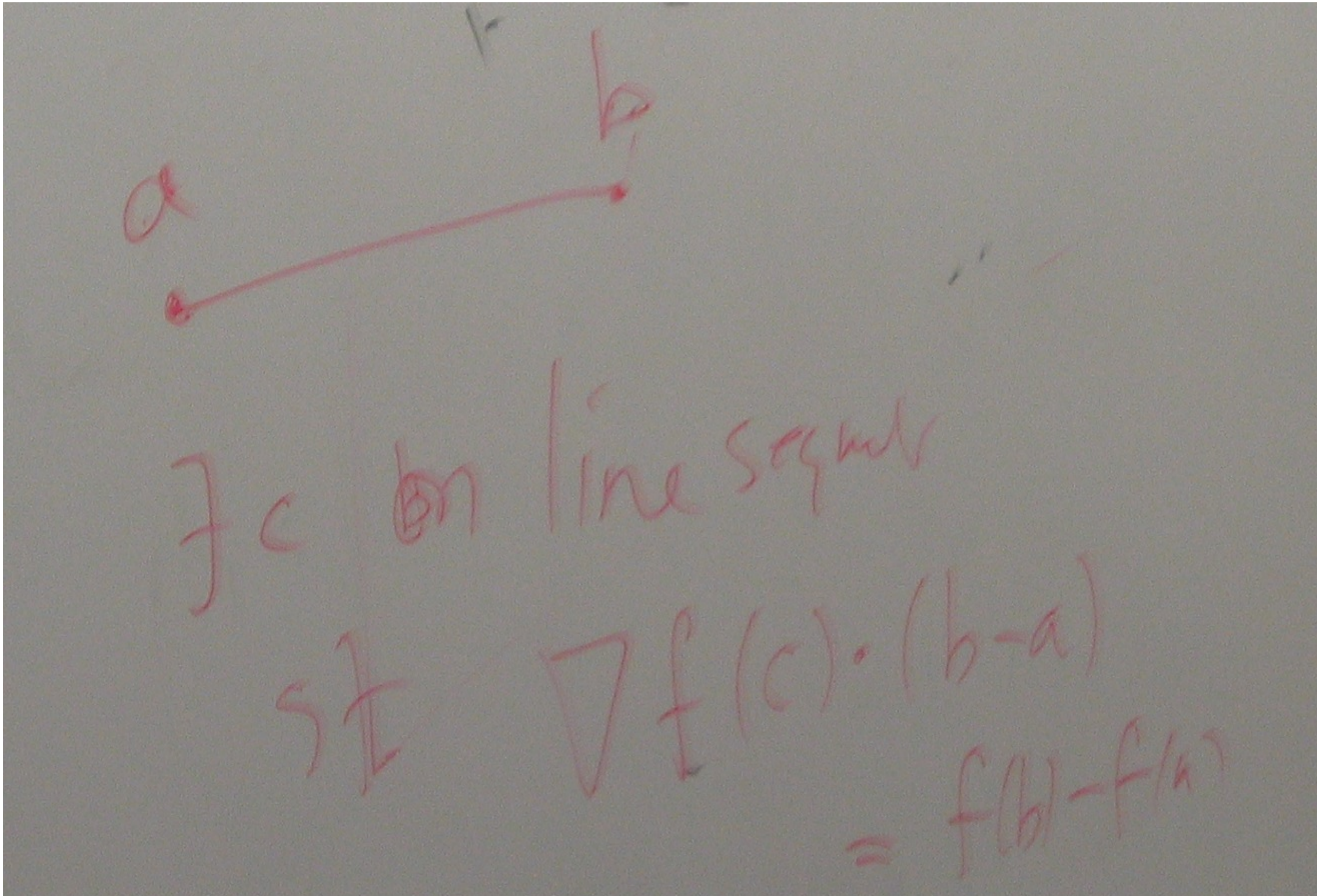

f'(c) = f(b)-f(a) / b-a

Gradient determines f up to a constant

- Let U be open and connected, and f and g be differentiable on U. If $\nabla f = \nabla g$ on U, then

  $$f(\mathbf{x}) = g(\mathbf{x}) + C.$$

- How would you prove this?
- Good enough to replace f by f-g and show that is $\nabla f = 0$ on U, then f is a constant.
- Connect any two points by "polygonal path" and use M.V. Theorem.



Which = 0.

Gradient determines f up to a constant

- Let U be open and connected, and f and g be differentiable on U. If $\nabla f = \nabla f$ on U, then

  $f(\mathbf{x}) = g(\mathbf{x}) + C.$
- How about an example?
- Arctan(y/x) vs. Arccos$(x/(x^2+y^2)^{1/2})$

In[1]:= `{D[ArcTan[y / x], x], D[ArcTan[y / x], y]}`

Out[1]= $\left\{-\frac{y}{x^2\left(1+\frac{y^2}{x^2}\right)}, \frac{1}{x\left(1+\frac{y^2}{x^2}\right)}\right\}$

In[2]:= `{D[ArcCos[x / Sqrt[x ^2 + y ^2]], x], D[ArcCos[x / Sqrt[x ^2 + y ^2]], y]}`

Out[2]= $\left\{-\frac{-\frac{x^2}{(x^2+y^2)^{3/2}}+\frac{1}{\sqrt{x^2+y^2}}}{\sqrt{1-\frac{x^2}{x^2+y^2}}}, \frac{x\,y}{(x^2+y^2)^{3/2}\sqrt{1-\frac{x^2}{x^2+y^2}}}\right\}$

In[3]:= `Simplify[%% - %]`

Out[3]= $\left\{-\frac{y}{x^2+y^2}+\frac{\sqrt{\frac{y^2}{x^2+y^2}}}{\sqrt{x^2+y^2}}, x\left(-\frac{y}{\sqrt{\frac{y^2}{x^2+y^2}}\,(x^2+y^2)^{3/2}}+\frac{1}{x^2+y^2}\right)\right\}$