The biggest, the smallest, the best, and the worst

T2_2401F08LavSols2.jpg (JPEG Image, 612x791 pixels)

I am only about halfway through at this point, but it seems like the biggest difficulty on page 2 was determining symmetry wrt coordinate planes. A lot of students tried to look at the projection onto the specified plane. Secondly, a lot of them missed that you were looking for an 'upward' normal and answered with its negative multiple. Thirdly, a small minority of the students still have problems taking partial derivatives, and their performance on page 2 suffered greatly as a result. That's the best analysis I can offer for now.

Mathematics 2401T **Test 2**

NAME: ______

4. Consider the surface correspond

a) Is this surface is symmetric about the xy plane ______ Not changed when $z \to -z$

The yz plane ___N___ (Y/N)? Because function changes when $x \to -x$.

b) Give a formula for the tangent plane to the surface at the point $(6, -3, 2)$.

ANSWER $(x-6) + 2(y+3) - 3(z-2) = 0$, or $x + 2y - 3z = -6$

① calculate a normal vector, which is any nonzero multiple of $\nabla(12x - 4y^2 - 9z^2) = 12\hat{i} - 8y\hat{j} - 18z\hat{k}$. At $(6,-3,2)$, $\vec{N} \propto 12\hat{i} + 24\hat{j} - 36\hat{k}$. might as well take $\frac{1}{12}$ of this to simplify.

5. (Dang, if this isn't just like an example done in class on Monday!) Consider the surface given as the graph $z = f(x, y)$ of the function $f(x, y) = y^2 - 3x^2$. The point $P = (1, -2, 1)$ is on this surface.

a) Find an upward normal vector to the surface at the point P.

ANSWER $6\hat{i} + 4\hat{j} + \hat{k}$

Can take $\vec{N} = -\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}$ At P.

$-\frac{\partial f}{\partial x} = 6x = 6$, $-\frac{\partial f}{\partial y}$

b) Give a formula for the norm

ANSWER $\{(x, y, z) =$

One other diagnostic observation: most students tried to calculate the directional derivative on the map problem by estimating a unit vector in the direction of the road and trying to dot that with the gradient vector. They didn't see that counting contour lines can also work in that situation.

Also there was some confusion about what type of equation or formula determines a plane vs. a line. So many students answered the tangent line question with some kind of plane equation, and to a lesser extent vice-versa.

6. On the attached topographic m near circled letters.

a) Draw arrows with bases at the of the gradient of the altitude functi

b) Estimate the magnitude of th are at heights differing by 20 feet and accuracy of about 10 % or better. ANSWER ______ feet.

c) If the gradient at any of the points on the graph H,N,Q,R is approximately **0**, list those points here: ______ H (top), Q (saddle) ______

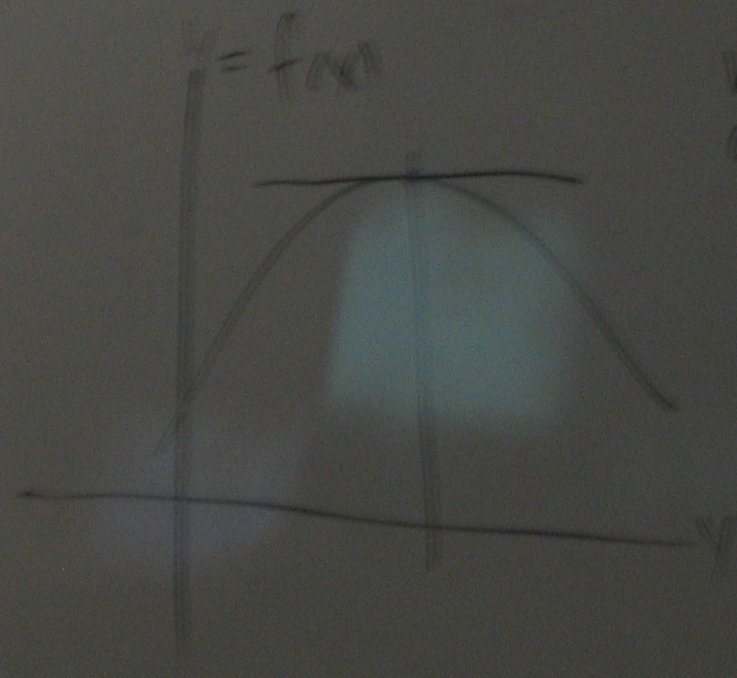
d) Estimate the directional derivative of the height function at the point P in the somewhat northerly direction along the road through P

Done

$z = f(x,y)$

A normal: $\left(-\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}\right)$

vertical component is $1 > 0$



One possibility is

max

1st find the value of x where $f'(x) = 0$

2nd substitute into $f(x)$

Maxima and minima

- What is the maximum value of

  $f(x,y) = 2 + 4\,x + x\,y - 3\,x^2 - 2\,y^2$?

If we imagine we know the best value of y, then the best value of x solves

$$0 = \partial f/\partial x = 4 + y - 6\,x.$$

Likewise, if we have the best x, then the best y solves

$$0 = \partial f/\partial y = x - 4\,y.$$

Maxima and minima

- Find the solution of the simultaneous equations:

  $0 = \partial f/\partial x = 4 + y - 6x.$

  $0 = \partial f/\partial y = x - 4y.$

Answer: $x = 16/23$, $y = 4/23$

The maximum value of $f(x,y)$ is $f(16/23, 4/23) = 78/23$.

It was the best of times, it was the worst of times.

- *And that’s a theorem*!

Absolute max and min

- $f(x_0) \geq f(x)$ for all x in D.

- THE THEOREM: If f is a _*continuous*_ function on a _*closed*, *bounded*_ set D, then f takes on an absolute maximum on D. Also an absolute minimum.

Local max or min

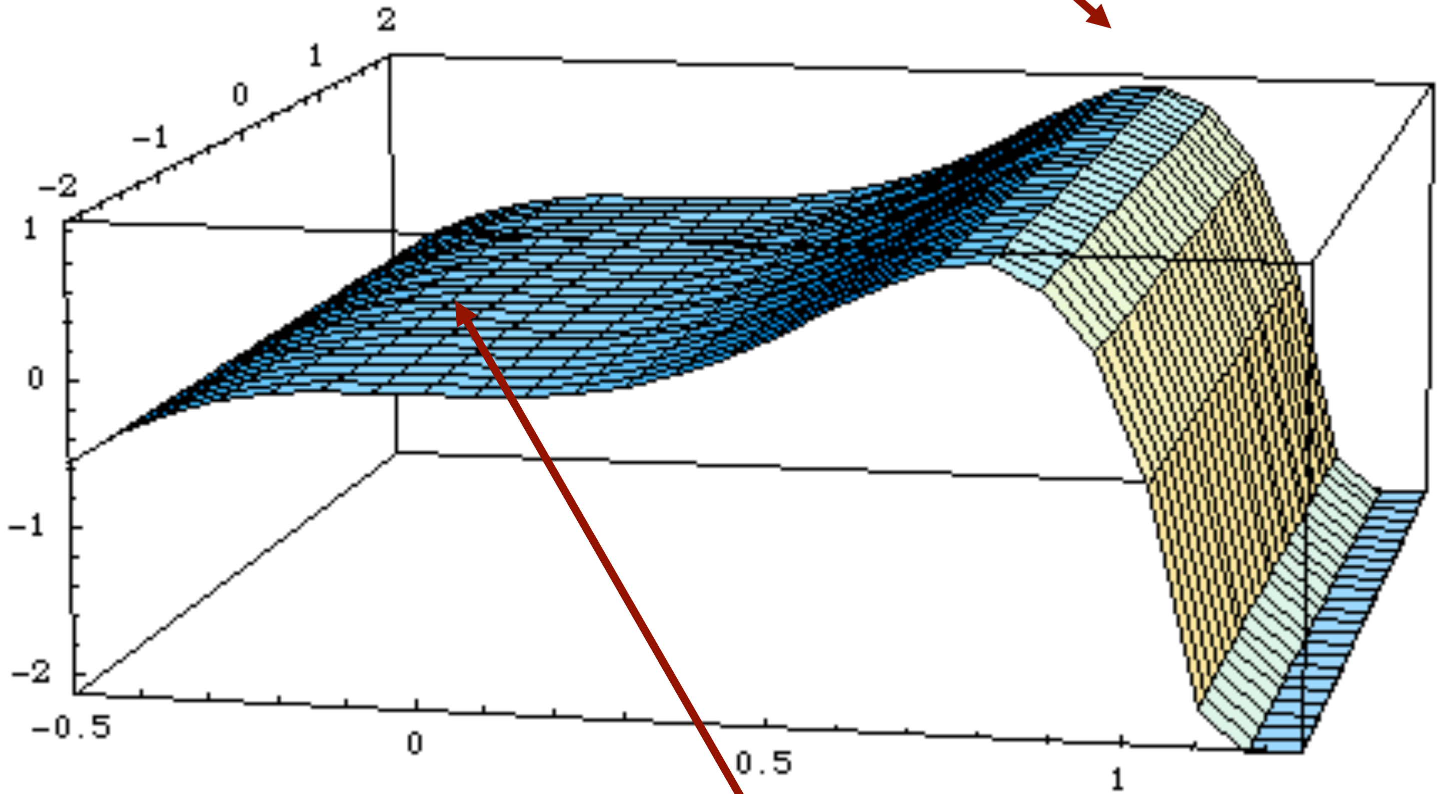
- Takes place at a *critical point*
  - Gradient = 0 (all components) OR
  - Gradient undefined

Maxima and minima- Discussion

- Let U be ______________ domain.
  - Important condition - connected
  - For *local* max/mins, assume open
- What is the condition for a local maximum or minimum?
  - It's still going to be a max if we vary x with y fixed, or vice versa. So the first partials must "vanish" (jargon for "= 0") or not exist. (crit pt)
  - That's *necessary*. Is it also sufficient?

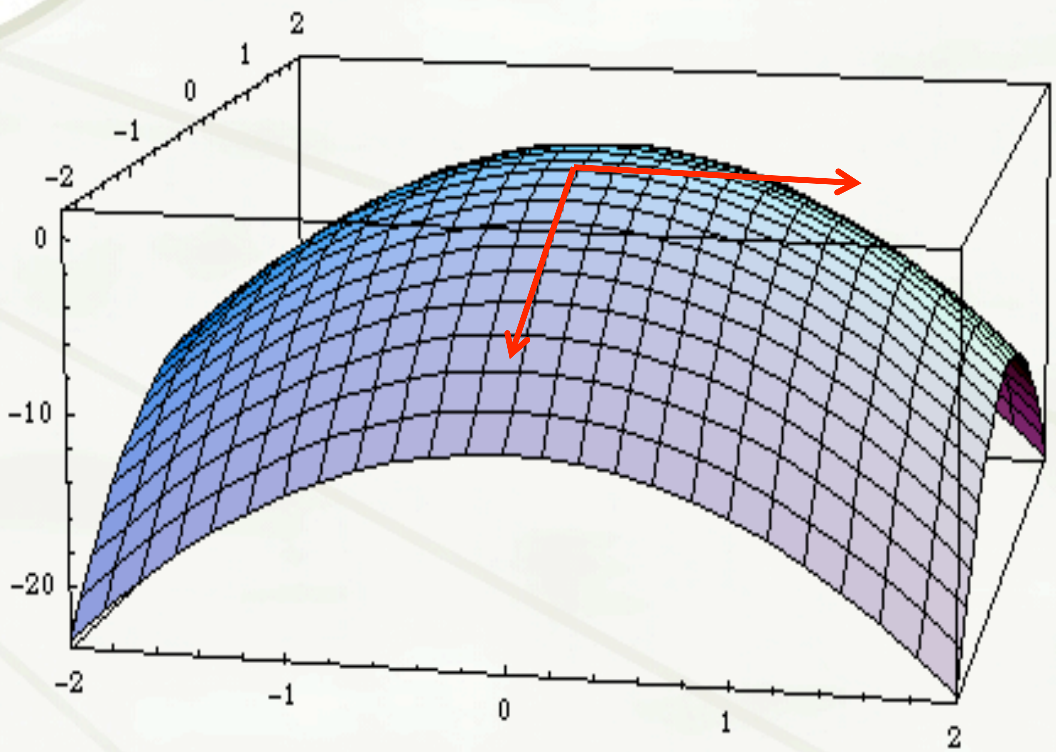
LOCAL MAX

```
In[12]:= Plot3D[4 x^3 - 4 x^6, {x, -1/2, 5/4}, {y, -2, 2},
  ViewPoint → {1, -5, 1}]
```



INFLECTION PT

In[4]:= Plot3D[1 - 3 x^2 - 3 y^2, {x, -2, 2}, {y, -2, 2}, ViewPoint → {1, -5, 1}]



Example

- What is the maximum value of

$$1 - 4\,(x^2 + y^2)^{1/2} = 1 - 4\,|\mathbf{r}|\ ?$$

Without calculus we can see it is 1, when **r** = **0**. But what if we differentiate?

Example

- What is the maximum value of

$$1 - 4\,(x^2 + y^2)^{1/2} = 1 - 4\,|\mathbf{r}|\ ?$$

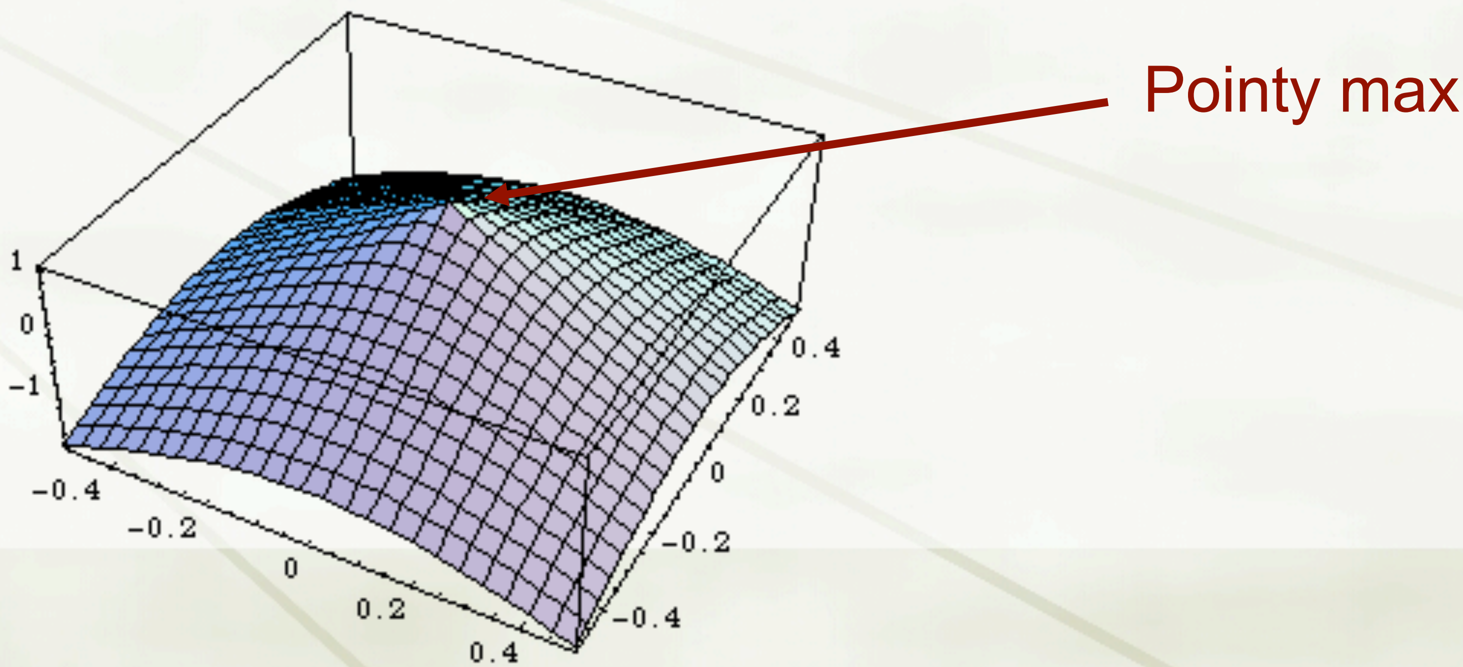
$$\nabla f(x,y) = -4\,(x/(x^2 + y^2)^{1/2})\,\mathbf{i} - 4\,(y/(x^2 + y^2)^{1/2})\,\mathbf{j}$$

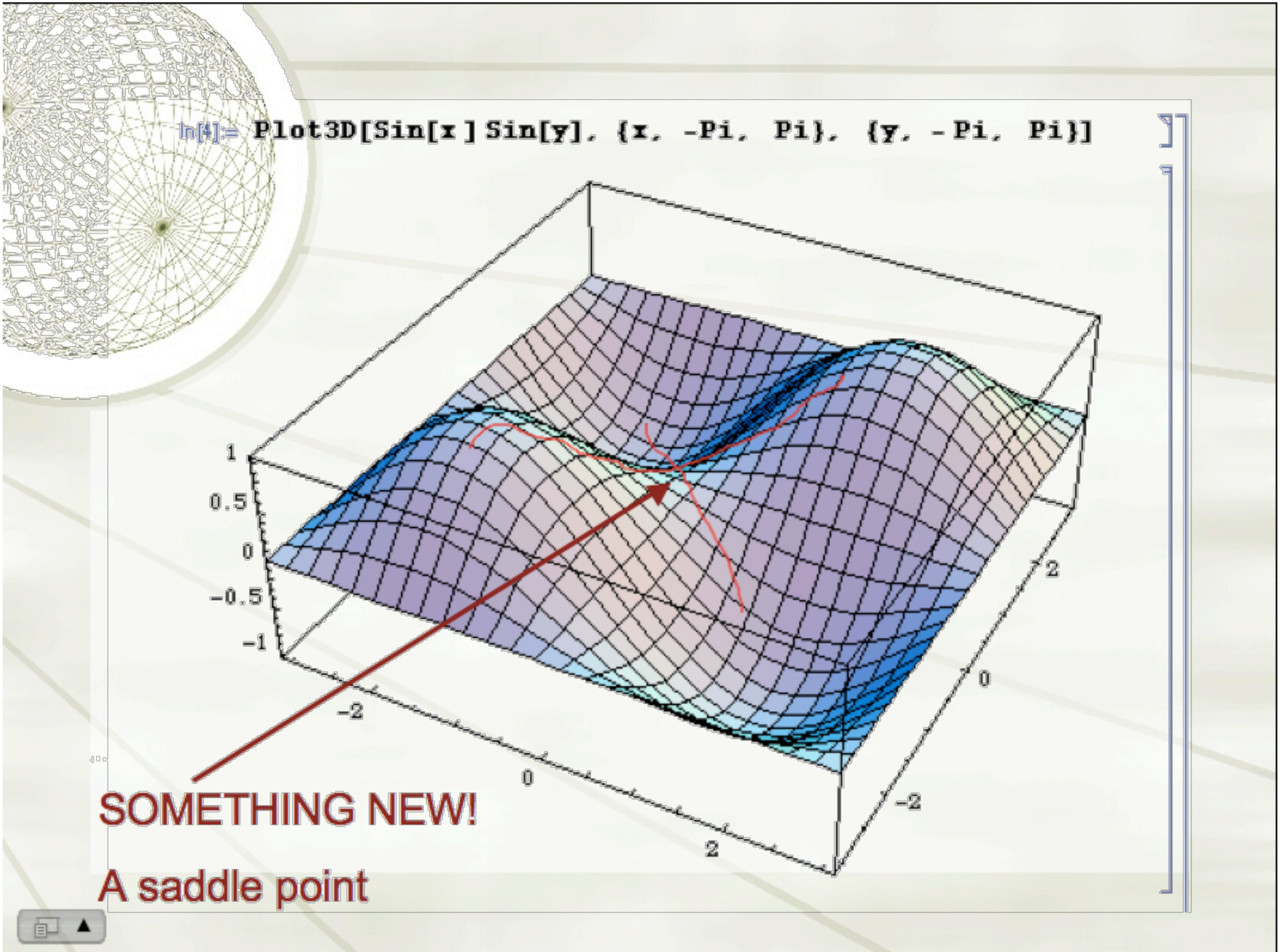
Example

- What is the maximum value of

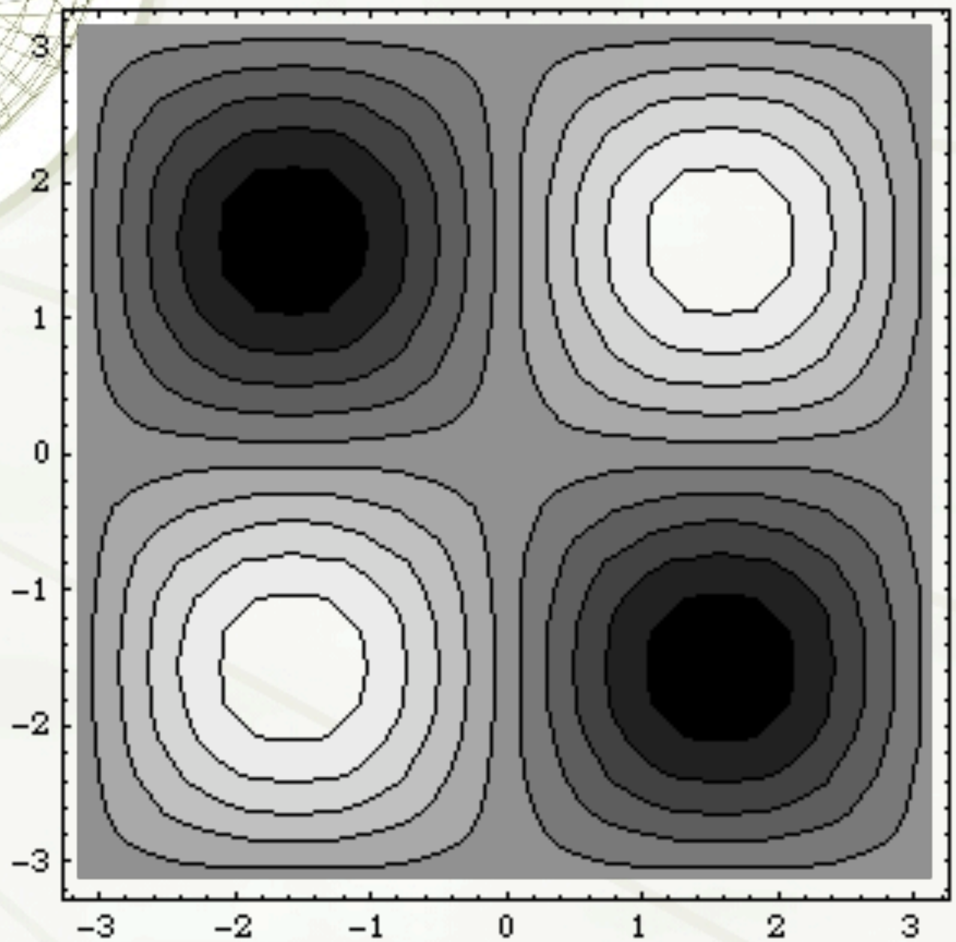
$$1 - 4\,(x^2 + y^2)^{1/2} = 1 - 4\,|\mathbf{r}|\ ?$$

```
In[22]:= Plot3D[1 - 4 Sqrt[x^2 + y^2], {x, -1/2, 1/2}, {y, -1/2, 1/2}]
```




In[4]:= Plot3D[Sin[x] Sin[y], {x, -Pi, Pi}, {y, -Pi, Pi}]
1
0.5
0
-0.5
-1
-2
0
2
2
0
-2
SOMETHING NEW!
A saddle point

In[15]:= ContourPlot[Sin[x] Sin[y], {x, -Pi, Pi}, {y, -Pi, Pi}]



Example

- What is the maximum value of

$$2 + 4x + xy - 3x^2 - 2y^2\ ?$$

Second derivative test?

- Recall 1D: $f''(x) > 0 \Rightarrow$ local minimum,

  $f''(x) < 0 \Rightarrow$ local maximum

Second derivative test?

- The **Hessian** matrix

$$\begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

Second derivative test?

- The **Hessian** matrix

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

Named for Ludwig Otto Hesse (not Hermann Hesse).

Second derivative test?

- The Hessian matrix

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

A symmetric matrix

The Eigenvalue....

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

I'm back!

Second derivative test?

- The Hessian matrix

$$\begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

Theorem: If det(H) < 0, then the critical point is a saddle.

Second derivative test

- $D = \det(H) = f_{xx} f_{yy} - (f_{xy})^2$.
  If at a crit. pt, and
  - $D < 0$, then SADDLE.
  - $D > 0$, then LOCAL MAX OR MIN. Check f_{xx} or f_{yy} to determine which:

    $f_{xx} > 0 \Rightarrow$ min, $\quad < 0 \Rightarrow$ max

Example

- sin(x) sin(y)   Where are the critical points, and what are they?

Example

- sin(x) sin(y) Where are the critical points, and what are they?

$$\vec{0} = \begin{bmatrix} \cos x \sin y \\ \sin x \cos y \end{bmatrix}$$

many poss. $x = \frac{2n+1}{2}\pi \Rightarrow$ also $y = \frac{2n+1}{2}\pi$

OR $x = n\pi \Rightarrow y = n\pi$

Example

- sin(x) sin(y)   Where are the critical points, and what are they?

$$H = \begin{pmatrix} -\sin(x)\sin y & \cos x \cos y \\ \cos x \cos y & -\sin x \sin y \end{pmatrix}$$

@ $x = m\pi,\ y = n\pi$

$$\begin{pmatrix} 0 & (-1)^m(-1)^n \\ (-1)^m(-1)^n & 0 \end{pmatrix}$$

$D = -1$

Saddle!

Example

- What is the maximum value of
  $$2 + 4x + xy - 3x^2 - 2y^2 ?$$

Examples - Find and classify critical points

- $4x^3 + y^2 - 12x^2 - 36x$
- $x y e^{-2xy}$
- $x^3 + (x - y)^2$