It was the best of times, it was the worst of times

About the test

- Median was 80.
- As it is written:

Students' grades will depend on the following quantity:

```
(T1  + T2  + T3  + T4 + Q  + F  - min(T1..T4,F) )   + E + F/2
```

where the components of this formula correspond to the ingredients mentioned above, after scaling so that all of them except E = extra credit total have a common median of 70. The drop in the formula is the *only* mechanism for coping with personal events such as illness and family emergencies. **There will be no opportunities for make-up tests after the fact. In the event of an absence due to travel representing Georgia Tech, such as an intercollegiate sports competition, you must notify the professor at least two weeks in advance** to arrange an early test or other alternative. Otherwise, such absences will be treated as personal.

About the test

- Percentiles:
  - 90th: 97
  - 75th: 89
  - 50th: 80
  - 25th: 63 (*Seek help.*)

Range: 32 to 100

Absolute max and min

- $f(x_0) \geq f(x)$ for all x in D.

- THE THEOREM: If f is a _*continuous*_ function on a _*closed*, *bounded*_ set D, then f takes on an absolute maximum on D. Also an absolute minimum.

Local max or min

- Takes place at a *critical point*
  - Gradient = 0 (all components) OR
  - Gradient undefined

Second derivative test?

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

Ludwig Otto Hesse – the Determinator

Second derivative test

- $D = \det(H) = f_{xx} f_{yy} - (f_{xy})^2$.

  If at a crit. pt, and

  - $D < 0$, then SADDLE.
  - $D > 0$, then LOCAL MAX OR MIN. Check f_{xx} or f_{yy} to determine which:

    $f_{xx} > 0 \Rightarrow$ min, $\quad < 0 \Rightarrow$ max

Examples - Find and classify critical points

- $4x^3 + y^2 - 12x^2 - 36x$
- $x y e^{-2xy}$
- $x^3 + (x - y)^2$

Calc. ∇f

find $\vec{r}$ s.t. $\nabla f = \vec{0}$

find $\vec{r}$ s.t. ∇f not def →

Local classific.

saddl

Calculate Hessian classify. loca

loc

Absolute classif. Abs max → evaluate

*also check b

$\nabla f = \vec{0}$
∇f not def
→ make a list

classify. saddles
local max
loc min

Abs max → evaluate f (candidates for local max)
& also check boundary

Examples - Find and classify critical points

- $4x^3 + y^2 - 12x^2 - 36x$
  - (3, 0) is a local minimum (Hessian matrix has positives on diagonal and positive determinant.)
  - (-1,0) is a saddle. (Hessian matrix has negative determinant.)

Examples - Find and classify critical points

- $4x^3 + y^2 - 12x^2 - 36x$

$$\nabla f = \begin{bmatrix} 12x^2 - 24x - 36 \\ 2y \end{bmatrix}$$

Need $y = 0$

$x = 3, -1$

- $x\,y\,e^{-2xy}$
- $x^3 + (x - y)^2$

Example - x y e^{-2xy}

$$\nabla \left(xy e^{-2xy}\right) = \begin{bmatrix} y - 2xy^2 \\ x - 2x^2 y \end{bmatrix} e^{-2xy}$$

When is this $= \vec{0}$?

Possib. 1. $y = 0$ for 1st component.

Forces $x = 0$ for 2nd component.

Possib 2. $1 - 2xy = 0$ for 1st component.

$x = \frac{1}{2y}$. 2nd component becomes

$\frac{1}{2y} - 2 \cdot \frac{1}{4y^2} \cdot y = 0$ automatically

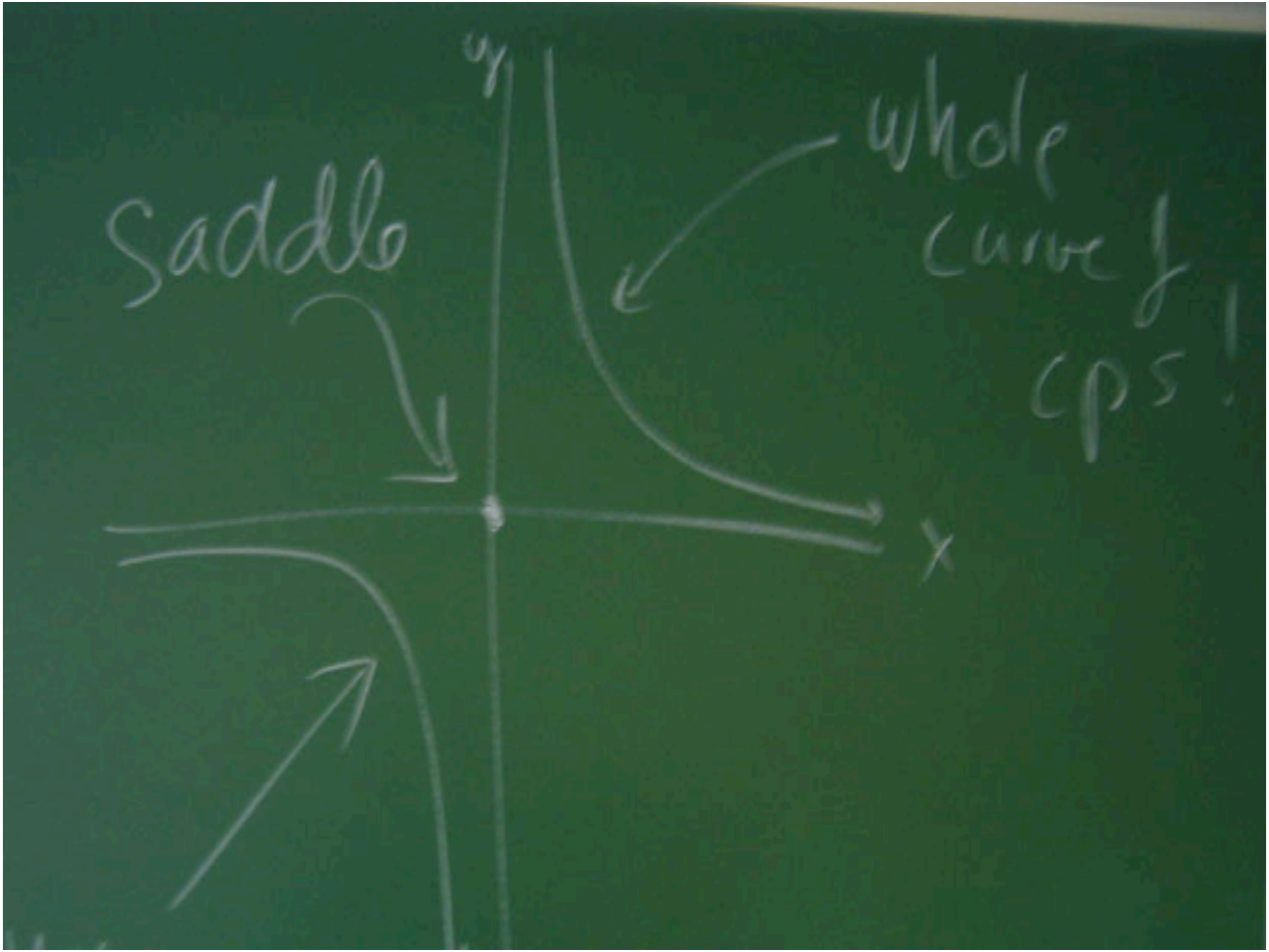
A whole curve of cp's!

Note:

$y - 2xy^2 = y(1 - 2xy)$

$$H = e^{-2xy}\begin{bmatrix} -4y^2+4xy^4 & 1-4xy+4x^2y^2 \\ 1-4xy+4x^2y^2 & -4x^2+4x^4y \end{bmatrix}$$

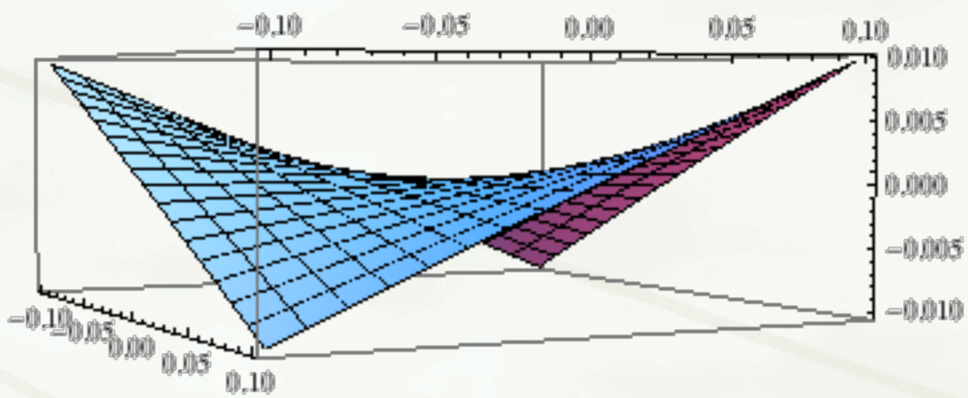
$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ at } \vec{0}.$$



saddle
whole curve } cps!
y
x

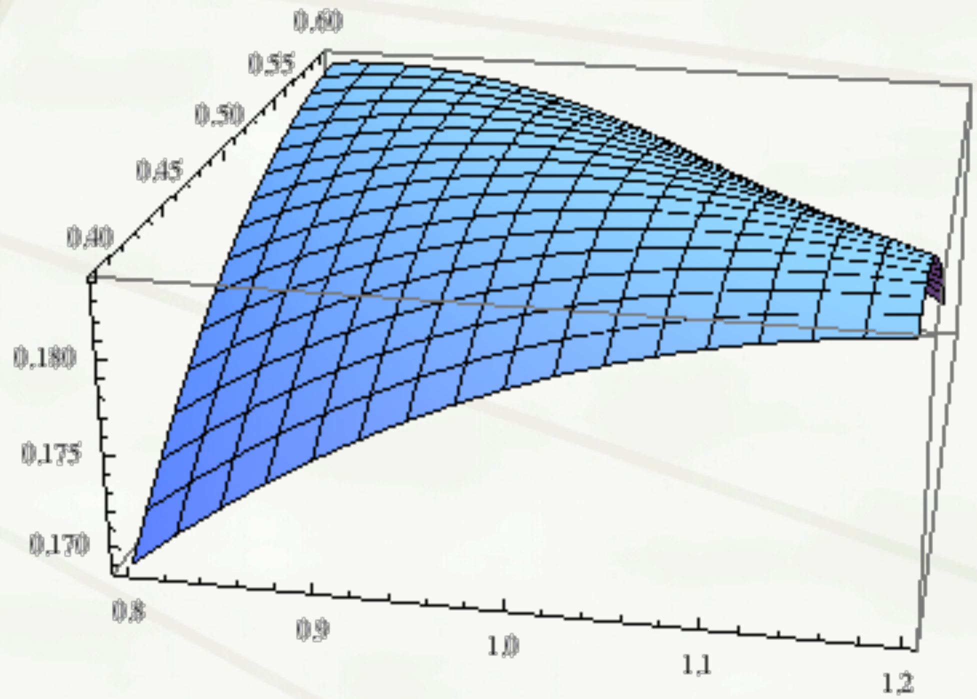
```
In[1]:= F[x_, y_] := x y E^(-2 x y)

In[8]:= Plot3D[F[x, y], {x, -.1, .1}, {y, -.1, .1}]
```

Out[8]=

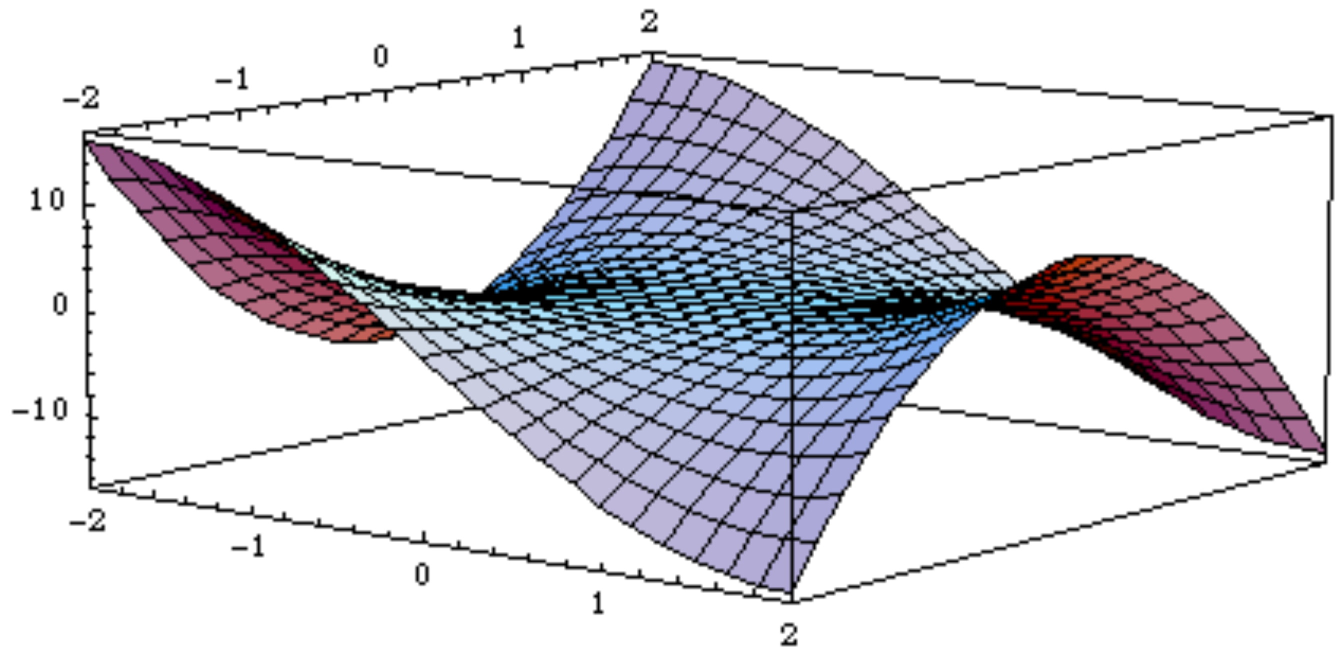


```
In[7]:= Plot3D[F[x, y], {x, .8, 1.2}, {y, .4, .6}]
```

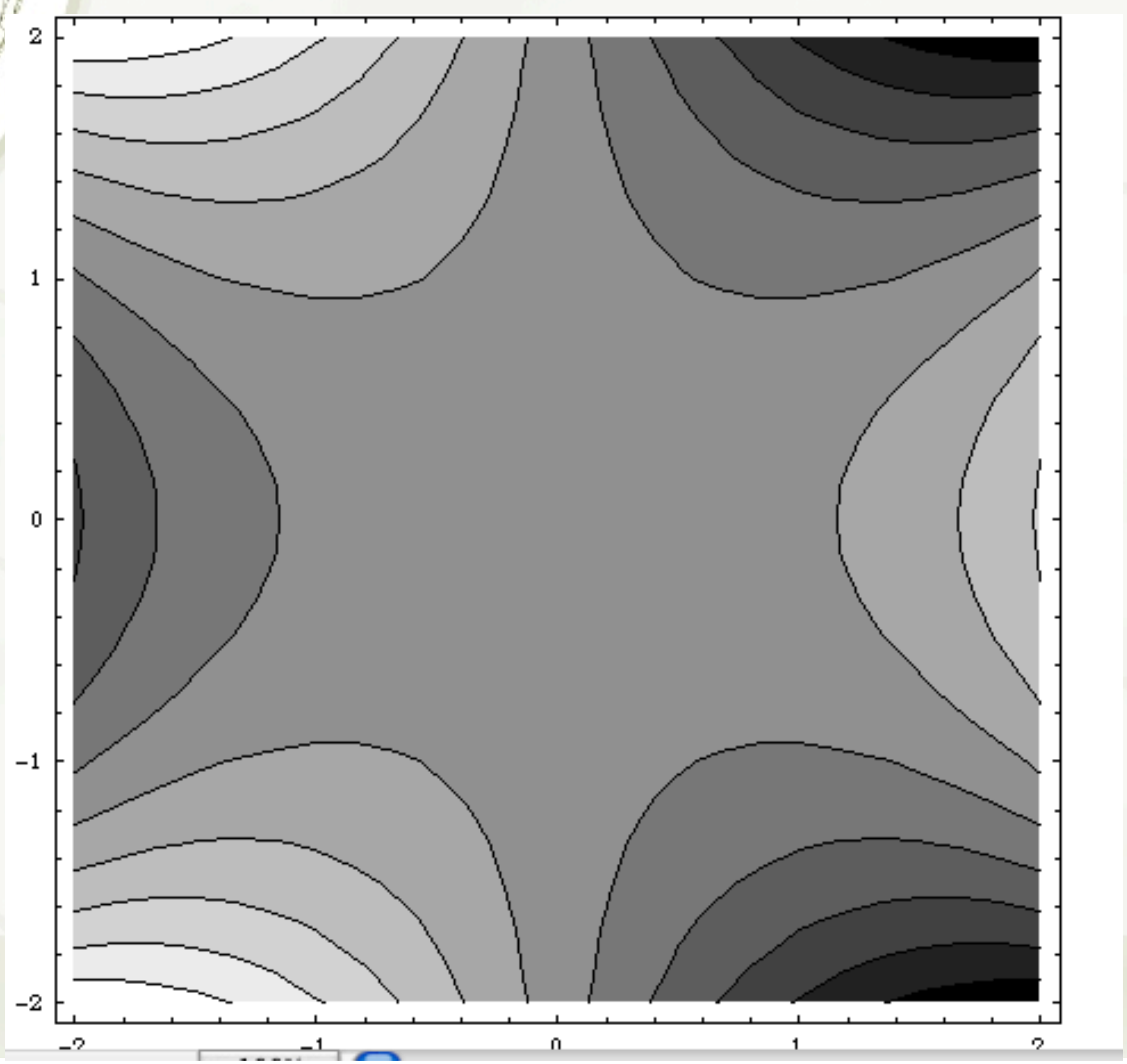
Out[7]=



A funky example



A funky example



Extra credit contest due next week

Due Thursday, 9 October. This contest has to do with the "funky example.” Find a formula for a saddle like that one. What do the gradient and Hessian matrix tell you about the funky saddle point at the origin x = y = 0? Carefully discuss the tangent planes and the (3D) normal vectors at points near the origin.

Second derivative test?

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}$$

Guess the theorem for 3D!

Second derivative test

- Find the eigenvalues of the Hessian matrix at a critical point.
  - If all of them are < 0, then LOCAL MAX
  - If all of them are < 0, then LOCAL MAX
  - Is some are < 0 and some are > 0, then the critical point is something like a saddle.
  - If one or more eigenvalues are 0, inconclusive.

Examples

- $f(x,y,z) = x^2 + 4y^2 - 8y + 3z^2 + xy - yz + 2$
- $f(x,y,z) = x^2 - 4y^2 - 8y + 3z^2 + xy - yz + 2$
- $f(x,y,z) = 7 - x^2 - 4y^2 - 8y - 3z^2 + xy - yz$

Absolute max and min

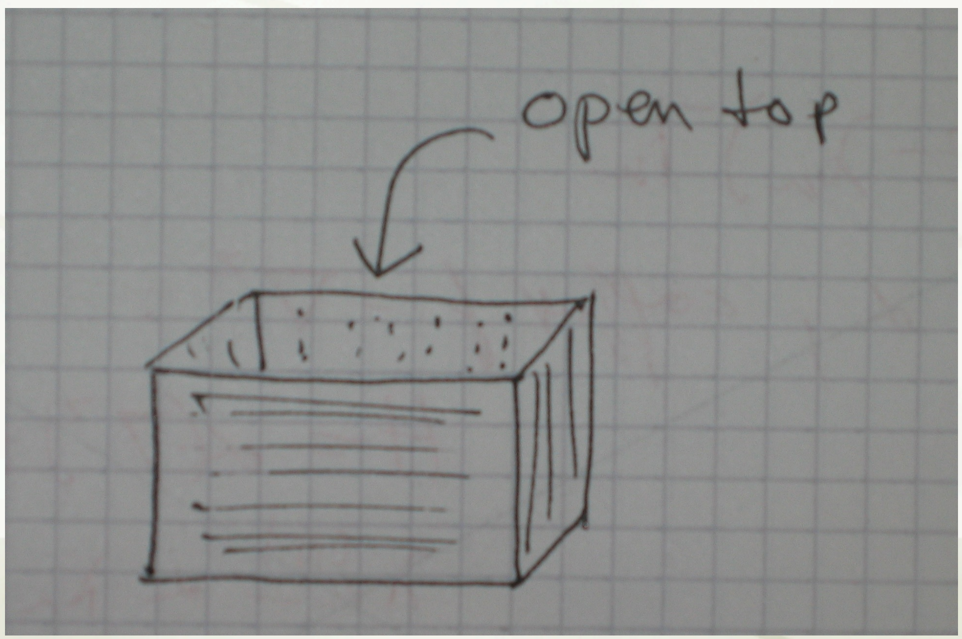
- $f(x_0) \geq f(x)$ for all x in D.
- THE THEOREM: If f is a _*continuous*_ function on a _*closed*, *bounded*_ set D, then f takes on an absolute maximum on D. Also an absolute minimum.

Absolute max and min

- Here is the algorithm for a closed, bounded region and continuous function:
  - Calculate gradient
  - List all critical points
    - Optional: Use Hessian test to eliminate some candidates.
  - Also check the boundary points
    - (more on this subject in future lecture)

A word problem – "modeling"

- How can you minimize the amount of wood needed to make a rectangular box, with no top, holding 1 cubic meter?



A word problem – “modeling”

- How can you minimize the amount of wood needed to make a rectangular box, with no top, holding 1 cubic meter?
- The *objective function* is $xy+2\,xz + 2\,yz$.
- The *constraints* are: $0 \le x, y, z$, and $xyz=1$.
- Use one constraint to eliminate one variable.

One option: Use the constraint to reduce the dimension. Substitute $z = 1/xy$ to get an objective function of the form

$$f(x,y) = x\,y + 2/x + 2/y\ .$$

Although this region is unbounded, we intuitively know that the box will have finite dimensions.

As for the constraints $0 \leq x,y,z$, we also know that on the boundary: $\{x=0\}$ or $\{y=0\}$ or $\{z=0\}$, the box would have 0 volume.

***The good values (x,y) are in the interior, and therefore are a critical point*.**

One option: Use the constraint to reduce the dimension. Substitute $z = 1/xy$ to get an objective function of the form

$f(x,y) = x y + 2/x + 2/y$.

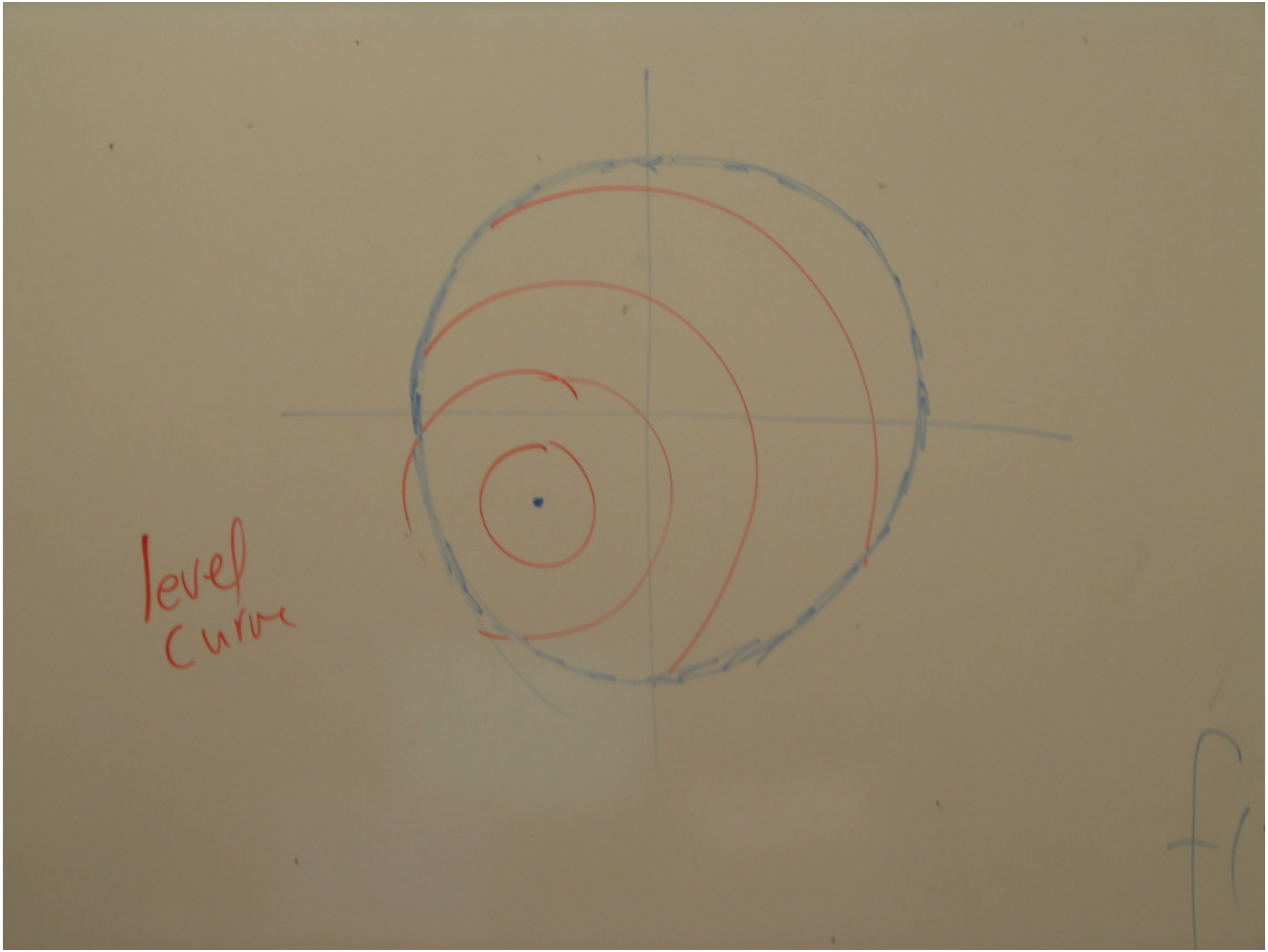
The gradient is

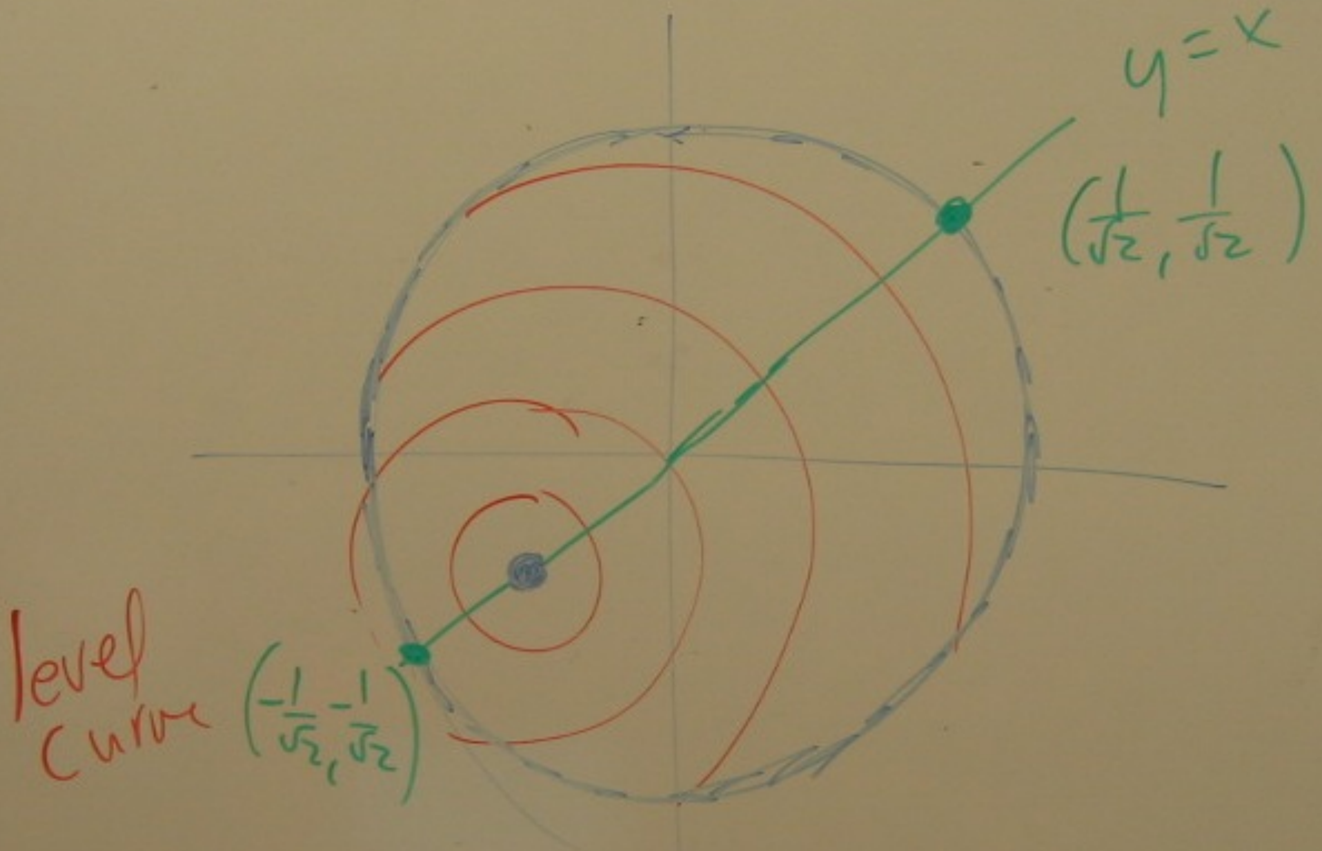
$$\nabla f(x,y) = (y - 2/x^2)\mathbf{i} + (x - 2/y^2)\mathbf{j}$$

- Solving gives $x = y = 2^{1/3}$ as the only critical point.
- z is then found to be $2^{-2/3}$.
- The amount of wood needed is $3 \cdot 2^{2/3} = 4.7622\ldots$ m

Example

- Let $f(x,y) = x^2 + y^2 + x + y$ on the closed unit disk. Find its absolute max and min.
  - Only 1 critical pt: $2x+1= 0 = 2y+1 \Rightarrow x=y=-1/2$.


level curve



$$f = x^2 + y^2 + x + y$$

on bdry $= \cos^2(t) + \sin^2(t) + \cos(t) + \sin(t)$

$f(x,y$

Example

- Note that $f(x,y) = x^2 + y^2 + x + y$ can also be written as

$$(x+½)^2 + (x+½)^2 - ½$$

(Complete the square.)

Example

- Let $f(x,y) = x^2 + y^2 + x + y$ on the closed unit disk. Find its absolute max and min.
  - Only 1 critical pt: $2x+1= 0 = 2y+1 \Rightarrow x=y=-1/2$.
  - On the boundary write $x = \cos t$, $y = \sin t$
  - f becomes $1+\cos t+\sin t$ and we are back in KG Calculus.

Example

- Let $f(x,y) = x^2 + y^2 + x + y$ on the closed unit disk. Find its absolute max and min.
  - The critical points of $1+\cos t+\sin t$ occur when $\cos t = \sin t$, I.e., $x = y$.
  - The boundary "candidates" for Max and Minnie are therefore $x = y = 2^{-1/2}$ and $x = y = - 2^{-1/2}$.

Example

- Plug the three candidates into $f(x,y) = x^2 + y^2 + x + y$:
  - The $f(-1/2, -1/2) = -1/2 = -0.5$

    Minnie!
  - $f(2^{-1/2},2^{-1/2}) = 1+ 2^{1/2} = 2.414214...$

    Max!
  - $f(-2^{-1/2},-2^{-1/2}) = 1- 2^{1/2} = -0.414214...$

    This candidate is a loser!

Constraints, regions, and side conditions

- These are all pretty much the same!
  - We can think of it as a region when x,y,z are position variables, even when the physical meaning is entirely different:
    - Temperature, pressure, volume
    - Cost of various items, sales