Making the best of it when you are up against the wall.

Math 2401 Assignments | Arthur Benjamin does "Mathem...

Mathematics 2401 **Calculus III** **Fall, 2008**

Current reading and homework assignments

- Due Thursday, 9 October:
  Reading:
  - SHE, Sections 16.7-17.2.
  - Review the lecture of 6 October
  - Review the lecture of 8 October
  - And, just for entertainment, look at some Mathemagic

  Exercises:
  - SHE, Section 16.7, # 2-4,10-12,15,24,27-30 (Same as last week. Hand in at least 5 even-numbered exercises.)
  - SHE, Section 16.8, # 3-6,14,18,21-24,29,30 (Same as last week. Hand in at least 5 even-numbered exercises.)
  - SHE, Section 16.9, # 2-6,10-12,17,18,26-28 (Hand in at least 5 even-numbered exercises.)

Current contests

Note about contest entries. These must be entirely your own work and not, for example, copied from the Web, even in modified form (which would be an honor code violation.

Unless otherwise specified, entries should be submitted to the professor, either in hard copy or by e-mail in a universally readable format, such as pdf.

1. Due Thursday, 9 October. This contest has to do with the "funky example" shown in class. Find a formula for a saddle like that one. What do the gradient and Hessian matrix tell you about the funky saddle point at the origin $x = y = 0$? Carefully discuss the tangent planes and the (3D) normal vectors at points near the origin.

Done

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A little-known tale from the 1001 Nights: Ali Baba and the shady gradient wallahs.

- *One day Ali Baba went to the bazaar, because he very much needed a function and its gradient. In the bazaar he met two gradient wallahs. The first of them came to him and spake thusly, " Esteemed sire, There is a wondrous function I will sell you for ten pieces of silver, and its gradient is*

$$\nabla f = (y\cos(xy) - y^2)\,\mathbf{i} + (x\sin(xy) - x^2 + 1)\,\mathbf{j}\,.$$ "

Ali Baba and the shady gradient wallahs.

- *Then came a second gradient wallah and tugged on Ali Baba's sleeve, proclaiming, "Sire, believe not yon notorious liar! Verily*

  $\nabla f = (y\cos(xy) - y^2)\,\mathbf{i} + (x\sin(xy) - x^2 + 1)\,\mathbf{j}$

  is bogus and counterfeit , and can in no wise be a gradient. But I can sell you a wondrous function for only 9 pieces of silver, and its true gradient is

  $\nabla f = (y\cos(xy) - y^2)\,\mathbf{i} + (x\cos(xy) - 2xy + 1)\,\mathbf{j}$. *"*

Ali Baba and the shady gradient wallahs.

- Whereupon the first merchant sprang at the second and a violent fight ensued, with entertaining clashes of scimitars and much blood and gore, whilst Ali Baba crouched against the wall, much with biting of the fingernails. Anon both merchants lay decapitated, and poor Ali Baba despaired of obtaining the function with the gradient. Then as usual his faithful and clever servant girl Morgiana appeared and saved Ali Baba's &%$@@(!, saying, "Oh. Master, do not despair, for one gradient wallah spake the truth, and I shall reveal both gradient and function!"

Which vector field is truly a gradient, and what is f?

$\nabla f = (y\cos(xy) - y^2)\,\mathbf{i} + (x\sin(xy) - x^2 + 1)\,\mathbf{j}$

$\nabla f = (y\cos(xy) - y^2)\,\mathbf{i} + (x\cos(xy) - 2xy + 1)\,\mathbf{j}$

$\nabla f = (y\cos(xy) - y^2)\,\mathbf{i} + (x\sin(xy) - x^2 + 1)\,\mathbf{j}$

$\nabla f = (y\cos(xy) - y^2)\,\mathbf{i} + (x\cos(xy) - 2xy + 1)\,\mathbf{j}$

In general,

$\nabla f = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$, where

$P(x,y) = f_x$ and $Q(x,y) = f_y$.

$\nabla f = (y\cos(xy) - y^2)\,\mathbf{i} + (x\sin(xy) - x^2 + 1)\,\mathbf{j}$

$\nabla f = (y\cos(xy) - y^2)\,\mathbf{i} + (x\cos(xy) - 2xy + 1)\,\mathbf{j}$

****Is $P_y = f_{yx}$ the same as $Q_x = f_{xy}$?****

$\nabla f_1 = (y\cos(xy) - y^2)\,\mathbf{i} + (x\sin(xy) - x^2 + 1)\,\mathbf{j}$

$\nabla f_2 = (y\cos(xy) - y^2)\,\mathbf{i} + (x\cos(xy) - 2xy + 1)\,\mathbf{j}$

****Is $P_y = f_{yx}$ the same as $Q_x = f_{xy}$?****

$$\partial P_1/\partial y = \cos(xy) - xy\sin(xy) - 2y$$

is *not* equal to

$$\partial Q_1/\partial x = \sin(xy) + xy\cos(xy) - 2x.$$

$\nabla f_1 = (y\cos(xy) - y^2)\,\mathbf{i} + (x\sin(xy) - x^2 + 1)\,\mathbf{j}$

$\nabla f_2 = (y\cos(xy) - y^2)\,\mathbf{i} + (x\cos(xy) - 2xy + 1)\,\mathbf{j}$

****Is $P_y = f_{yx}$ the same as $Q_x = f_{xy}$?****

Whereas

$$\partial P_2/\partial y = \cos(xy) - xy\sin(xy) - 2y = \partial Q_2/\partial x$$

as it should.

Ah, but what is f?

- We can integrate in x for fixed y or vice versa. The constant of integration if you "fix" y temporarily might vary when y is no longer held fixed. It is therefore a function of y - but not x.
- And conversely, switching x and y

$\nabla f = (y\cos(xy) - y^2)\,\mathbf{i} + (x\sin(xy) - x^2 + 1)\,\mathbf{j}$

$\nabla f = (y\cos(xy) - y^2)\,\mathbf{i} + (x\cos(xy) - 2xy + 1)\,\mathbf{j}$

****Is $P_y = f_{yx}$ the same as $Q_x = f_{xy}$?****

Do it both ways and compare!

$\int (y\cos(xy) - y^2)\,dx$ (y fixed)

$= \sin(xy) - y^2 x + C(y)$

$\int Q(x,y)\,dy = \sin(xy) - xy^2 + y + C_2(x)$ (x fixed)

$\nabla f = (y\cos(xy) - y^2)\,\mathbf{i} + (x\sin(xy) - x^2 + 1)\,\mathbf{j}$

$\nabla f = (y\cos(xy) - y^2)\,\mathbf{i} + (x\cos(xy) - 2xy + 1)\,\mathbf{j}$

****Is $P_y = f_{yx}$ the same as $Q_x = f_{xy}$?****

By comparing both calculations,

This one is an absolute constant

$f(x,y) = \sin(xy) - xy^2 + y + C$

An example from last week

- Let $f(x,y) = x^2 + y^2 + x + y$ on the closed unit disk. Find its absolute max and min.
- *Is there an easier way?*

Example

- Let $f(x,y) = x^2 + y^2 + x + y$ on the closed unit disk. Find its absolute max and min. Previous solution:
  - Only 1 critical pt: $2x+1= 0 = 2y+1 \Rightarrow x=y=-1/2$.
  - On the boundary write $x = \cos t$, $y = \sin t$
  - f becomes $1+\cos t+\sin t$ and we are back in KG Calculus.
  - Check 3 possibilities.

Example

- Plug the three candidates into $f(x,y) = x^2 + y^2 + x + y$:
  - The $f(-1/2, -1/2) = -1/2 = -0.5$ Minnie!
  - $f(2^{-1/2},2^{-1/2}) = 1+ 2^{1/2} = 2.414214\ldots$ Max!
  - $f(-2^{-1/2},-2^{-1/2}) = 1- 2^{1/2} = -0.414214\ldots$

This candidate is a loser!

Constraints, regions, and side conditions

- These are all pretty much the same!
  - We can think of it as a region when x,y,z are position variables, even when the physical meaning is entirely different:
    - Temperature, pressure, volume
    - Cost of various items, sales

Example

- Let $f(x,y) = x^3 y^4 z^5$ with the side conditions $x,y,z \geq 0$, $x+y+z \leq 1$.
- Point to ponder: What does the *feasible set* look like?

Example

- Let $f(x,y) = x^3 y^4 z^5$ with the side conditions $x,y,z \geq 0$, $x+y+z \leq 1$.

- Point to ponder: What does the *feasible set* look like?
  - A tetrahedron, a 4-sided figure each face of which is a triangle.
  - *Do you see this?*

Examples

- How big can the product of two numbers be if you fix the sum?
- How small?
- How big/small can the sum be if you fix the product?

Examples

- How big can the product of two numbers be if you fix the sum?
  - Objective function: $f(x,y) = xy$
  - Constraint: $x+y = 12$ (say)
  - Plug: $f \to x(12 - x)$. Maximize this with Calc I when $0 = 12 - 2\,x_{\bullet}$, so $x_{\bullet}=6$. Also $y_{\bullet}=6$.

Interpretation

- How big can the product of two numbers be if you fix the sum?
- This is the same as maximizing the area xy of a rectangle of sides x and y, when you fix the perimeter 2(x+y). It's at least plausible that the square is the best shape, so x=y.

Examples

- OK…. So how big can the product of *three* numbers be if you fix the sum?
  - Objective function: $f(x,y) = xyz$
  - Constraint: $x+y+z = 12$ (say)

Examples

- Or…. So how big can the product of *four* numbers be if you fix the sum?

Meet a great optimist

- er, optimizer, -

Joseph-Louis Lagrange

The Lagrange condition

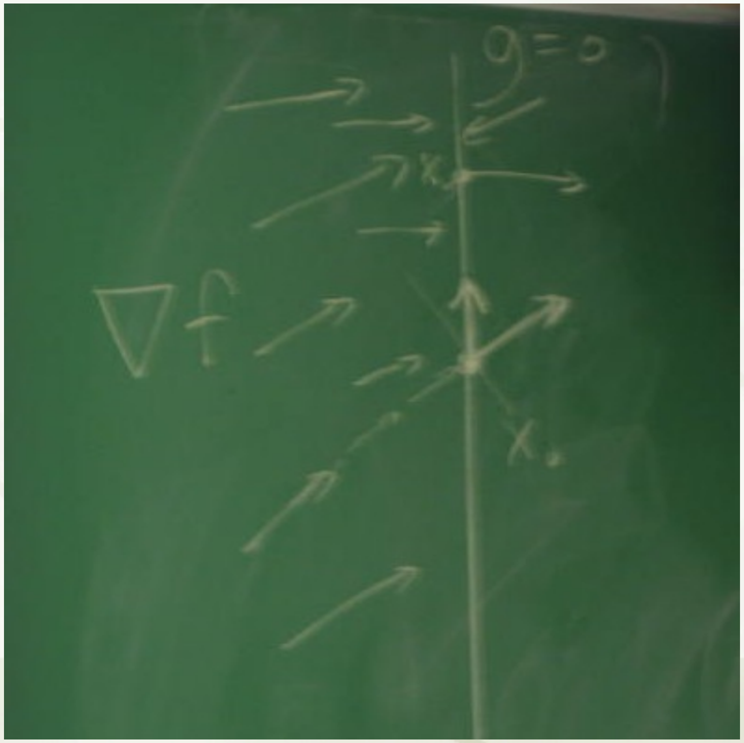
- A constrained max or min subject to $g(x,y,z) = 0$ satisfies…
  - Think: $f(\mathbf{r}(t))$ will be maximized as long as the curve $\mathbf{r}(t)$ stays on the curve or surface $C = \{\mathbf{r} : g(\mathbf{r}) = 0\}$.
  - What do we know about the scalar function $h(t) = f(\mathbf{r}(t))$ at max or min?

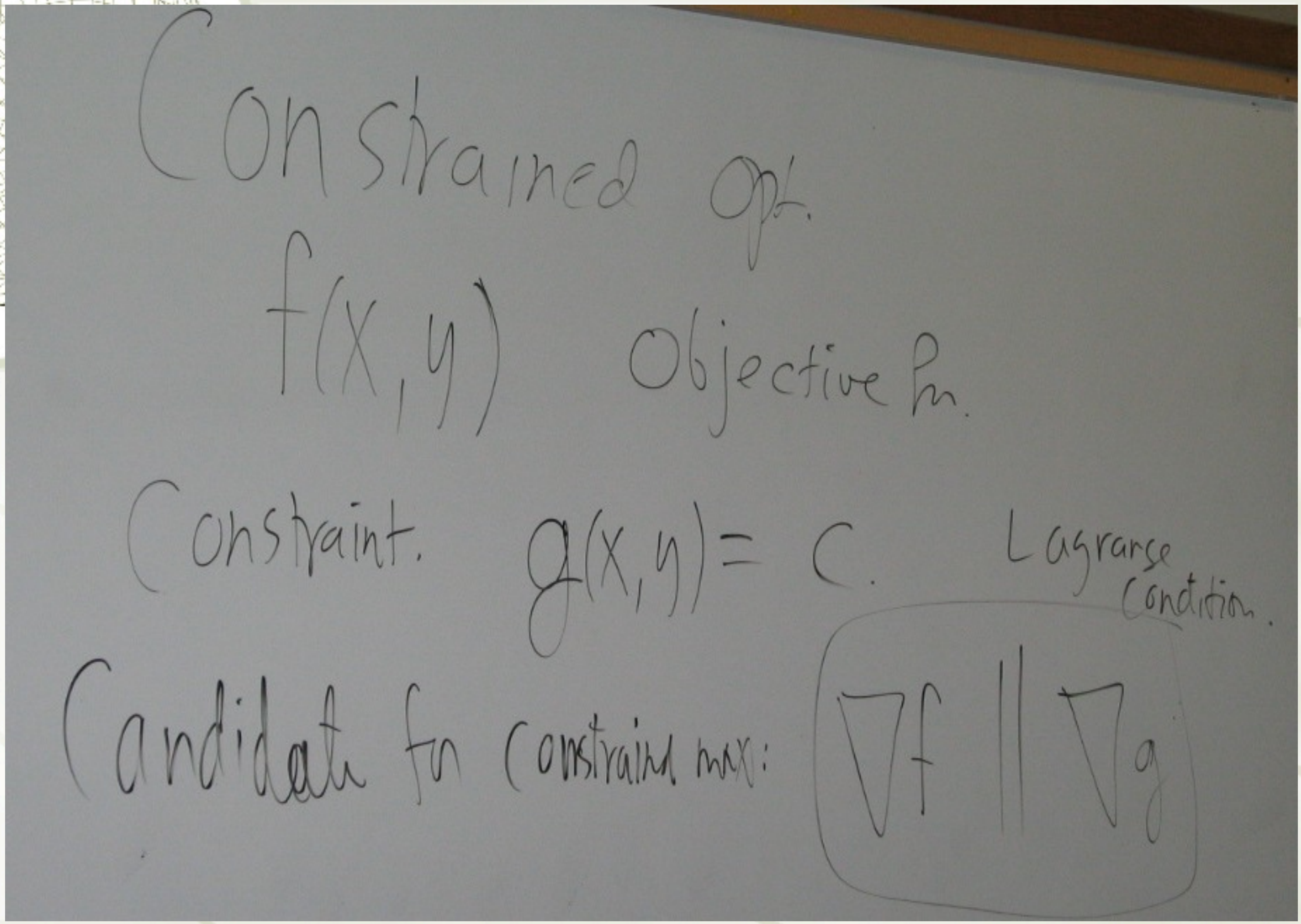
The Lagrange condition

- Ans: $0 = \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t)$, and the velocity $\mathbf{r}'(t)$ can have any direction parallel to C.
- *So* $\nabla f(\mathbf{r}(t))$ *has to be perpendicular to the surface or curve* $C = \{\mathbf{r} : g(\mathbf{r}) = 0\}$.
- Don't we know another gradient that is perpendicular to this level set of the function g?
  - *Sure.* ∇g *is also perpendicular to the level sets of* g. *So* ∇f *and* ∇g *must be parallel.*

The Lagrange condition

- Here is a pictorial understanding of the condition:




Constrained opt.
f(x, y) Objective fn.
Constraint. g(x, y) = c.
Lagrange condition.
Candidate for constrained max: ∇f || ∇g

The Lagrange condition

- Assuming f and C are “smooth,” at a boundary maximum point $\mathbf{x}_0$ where $\nabla g(\mathbf{x}_0) \neq \mathbf{0}$,

$$\nabla f(\mathbf{x}_0) = \lambda \, \nabla g(\mathbf{x}_0)$$

for some scalar value λ.

Objective

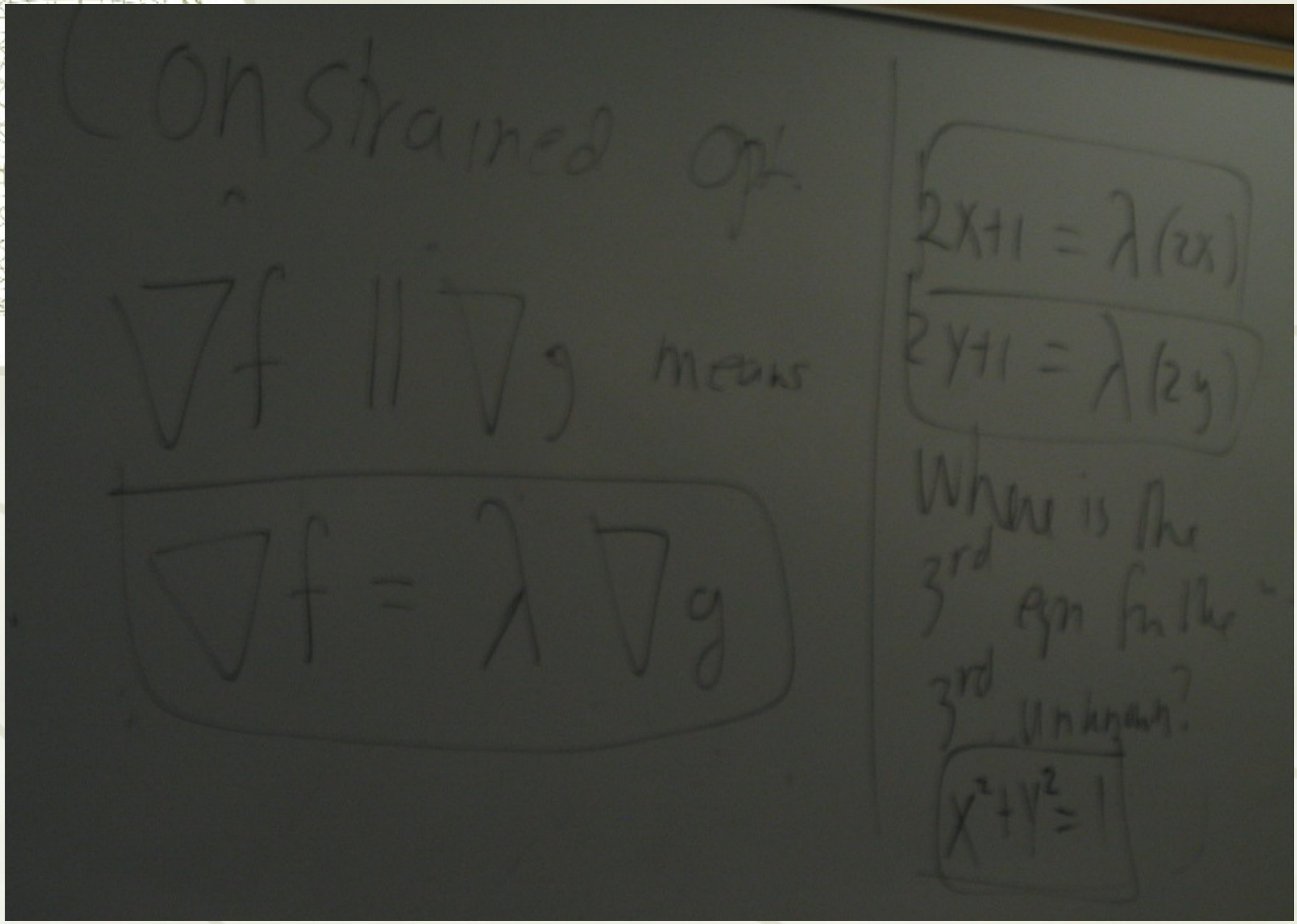
$$f(x,y) = x^2 + y^2 + x + y$$

$$\nabla f = (2x+1)\hat{i} + (2y+1)\hat{j}$$

Constraint: boundary is a level curve {

$$g(x,y) = x^2 + y^2 \; (= 1)$$

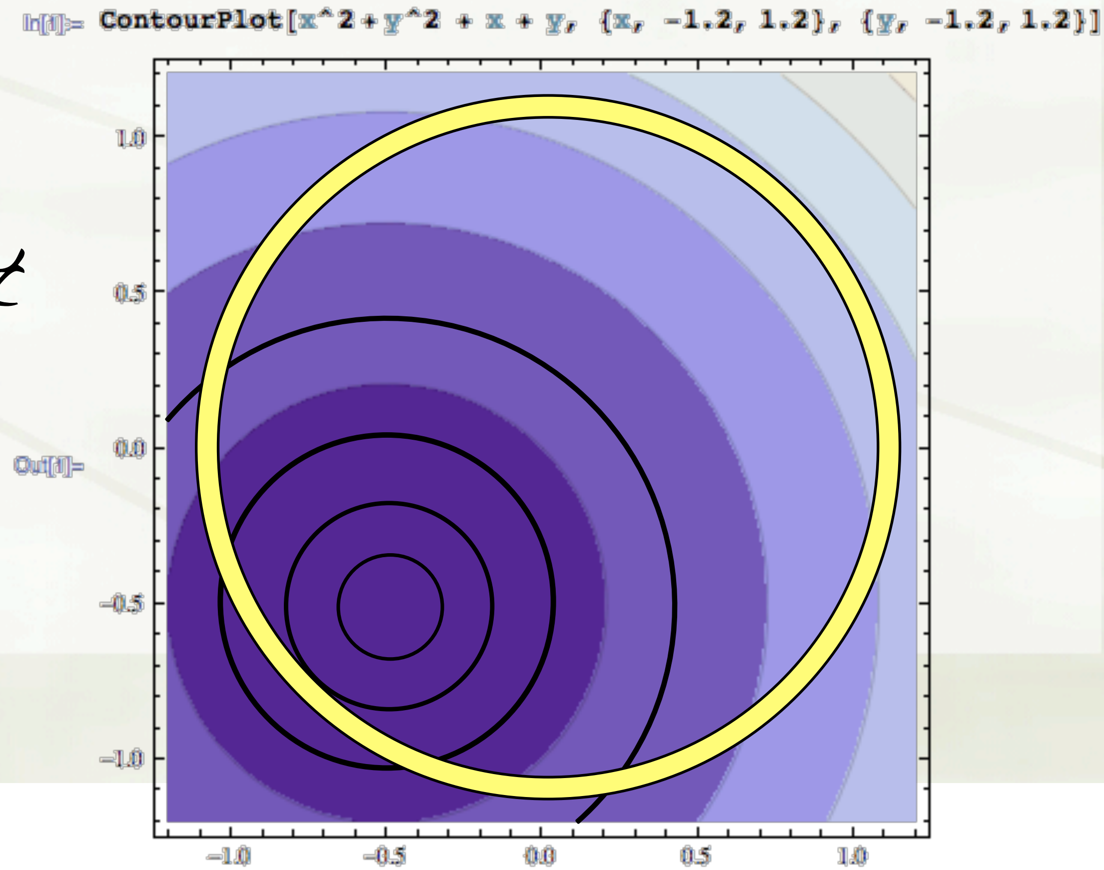
$$\nabla g = 2x\hat{i} + 2y\hat{j}$$


Constrained opt.
∇f || ∇g means
∇f = λ ∇g
2x+1 = λ(2x)
2y+1 = λ(2y)
Where is the 3rd eqn for the 3rd unknown?
x²+y²=1

An example from last week

- Let $f(x,y) = x^2 + y^2 + x + y$ on the closed unit disk. Find its absolute max and min.

Let's look at level curves:



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- Let $f(x,y) = x^2 + y^2 + x + y$ on the closed unit disk. Find its absolute max and min.

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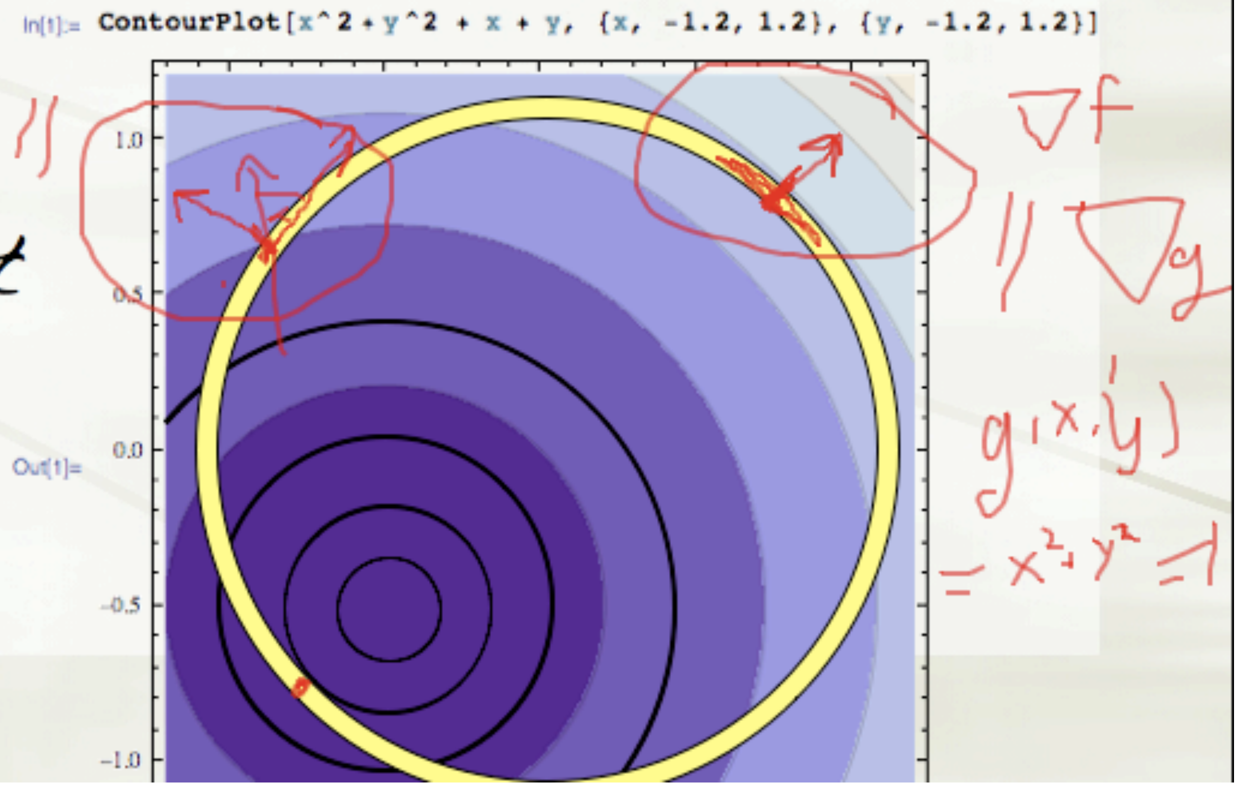

Examples

- Closest point to the origin of the plane $x + 2y - 3z = 6$.

- Extrema of $2x + y$, assuming $x^2 + 2y^2 = 18$.

Examples

- Closest point to the origin of the plane $x + 2y - 3z = 6$.
  - $f(\mathbf{r})$ could be $x^2 + y^2 + z^2$, rather than the distance. This avoids differentiating square roots, and it is just as good for answering the question.
  - $g(\mathbf{r}) := x + 2y - 3z - 6$

Examples

- Closest point to the origin of the plane

$g(\vec{r}) =$ x + 2 y - 3 z = 6.

$f = (\text{dist from } \vec{O})^2 = x^2 + y^2 + z^2$

$\nabla f = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \rightarrow y = 2x, \; z = -3x$

$x + 2y - 3z = 6$

$6 = x + 4x + 9x$

$x = 3/7$

- Extrema of 2 x + y, assuming

$x^2 + 2\,y^2 = 18$.

Examples

- Closest point to the origin of the plane $x + 2y - 3z = 6$.
- Extrema of $2x + y$, assuming $x^2 + 2y^2 = 18$.

Examples

- How big can the product be on the unit circle?
- On the square $0 \leq x,y \leq 1$?

Examples

Lagrange this time

- How big can the product of two numbers be if you fix the sum?

$f(x,y) = xy \qquad g(x,y) = x+y$

- How small?

$$\nabla f = \begin{bmatrix} y \\ x \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- How big/small can the sum be if you fix the product?