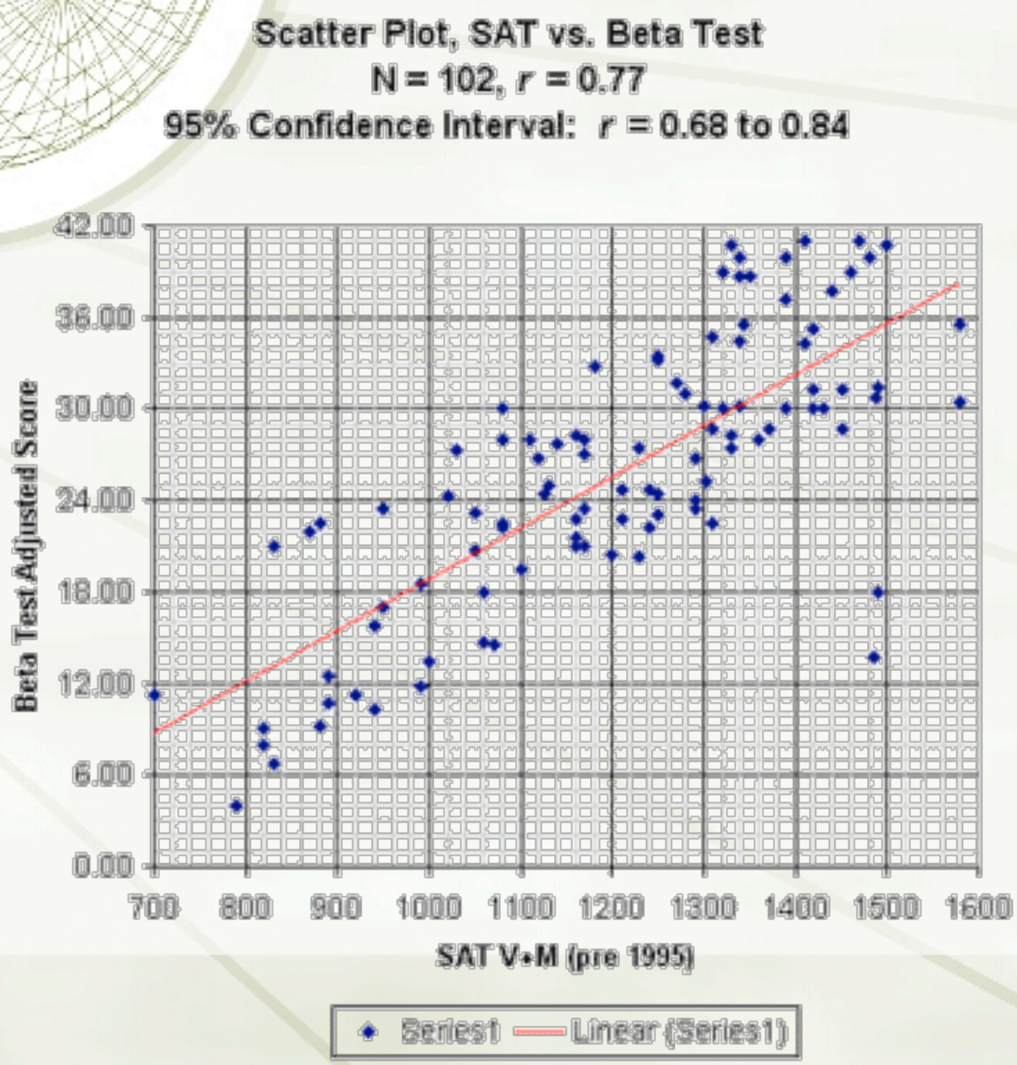
Part One, or

If you must be wrong, how little wrong can you be?

How does the Army beta test correlate with the SAT?



Source: army.mil

Error analysis and optimization

- You have a bucket of data $\{(x_i, y_i)\}$. They don't really fit on a line, but what is the best fit in the *least-squares sense*?
- You want to minimize

  $$\sum_i (y_i-(mx_i+b))^2$$

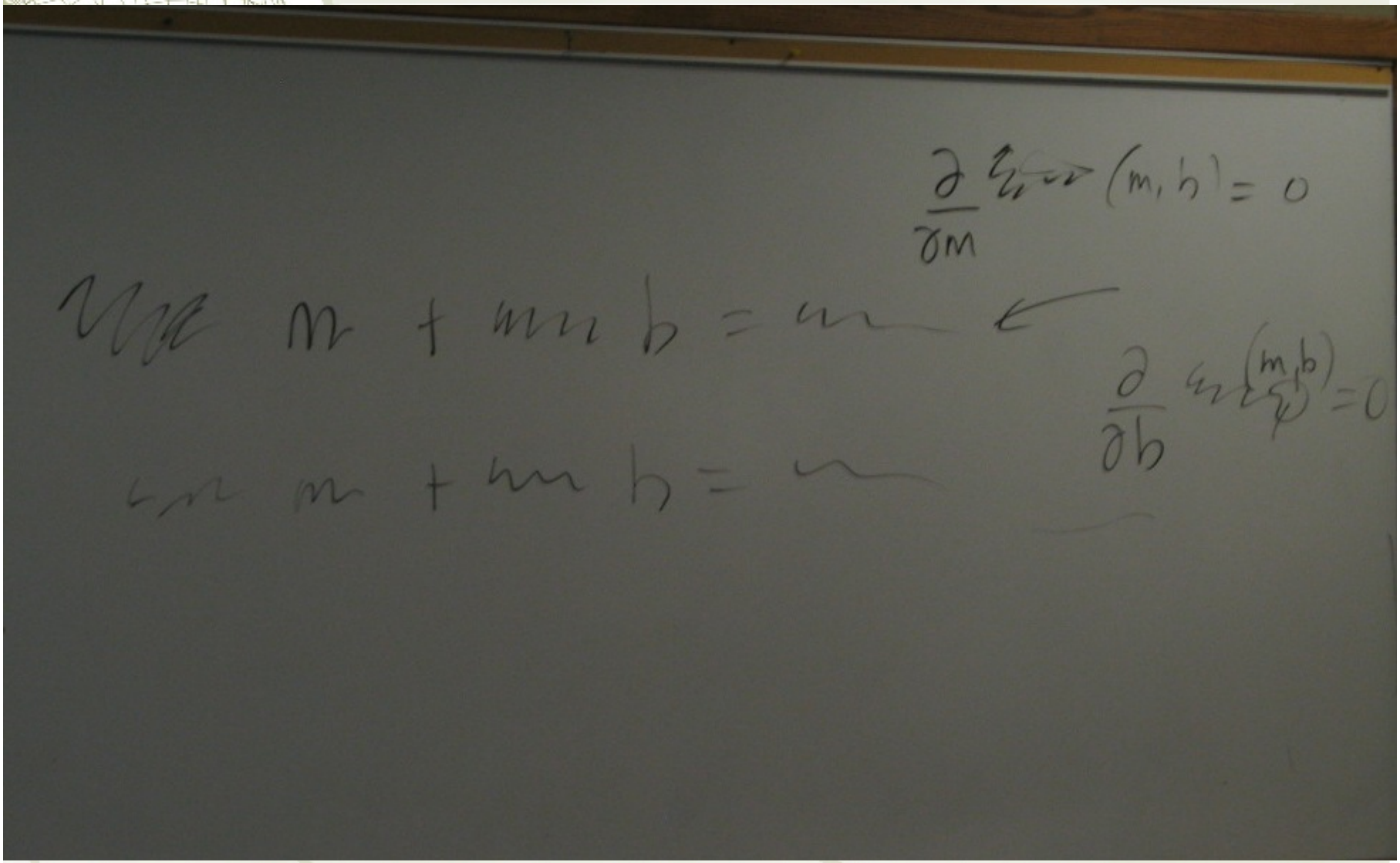
- But what are the %(@!! variables?
  - ANSWER: m and b !

Error analysis and optimization

- So we take the gradient of the objective function

  $f(m,b) := \sum_i (y_i - (mx_i + b))^2$

  with respect to variables m and b !
- Critical points when
  - $0 = -2 \sum_i x_i (y_i - (mx_i + b))$ and
  - $0 = -2 \sum_i (y_i - (mx_i + b))$


(m, b) = 0
(m,b)
=0

Error analysis and optimization

- Rewrite
  - $0 = -2 \sum_i x_i(y_i-(mx_i+b))$ and
  - $0 = -2 \sum_i (y_i-(mx_i+b))$
- In the form of a linear system of equations like ____m + ____ b = ____:

$$(\sum_i x_i^2)m + (\sum_i x_i)\, b = \sum_i x_i y_i$$

$$(\sum_i x_i)m + N\, b = \sum_i y_i$$

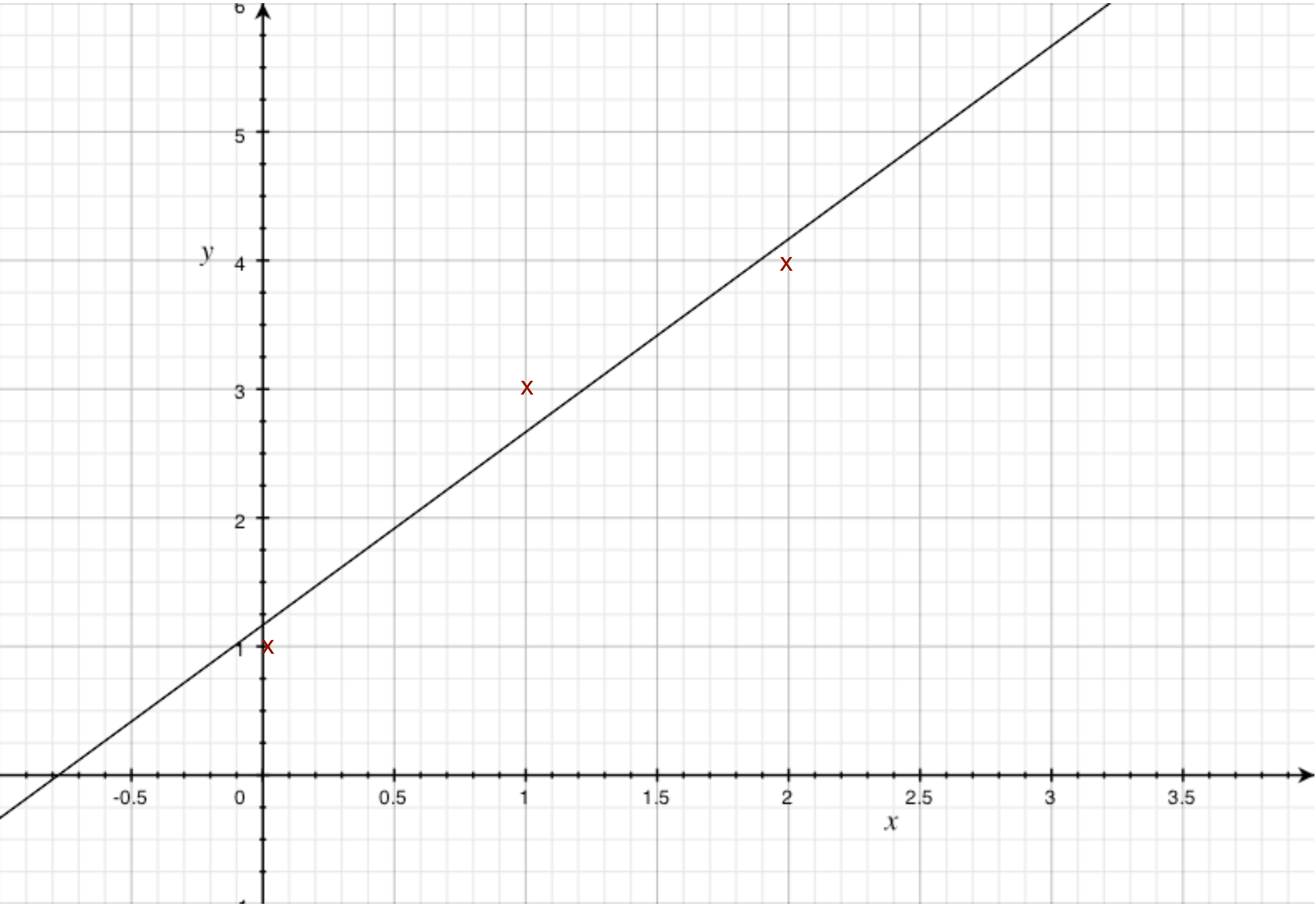
$$\begin{bmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & N \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \sum x_i y_i \\ \sum y_i \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} m \\ b \end{bmatrix} = \frac{1}{N\sum x_i^2 - (\sum x_i)^2} \begin{bmatrix} N & -\sum x_i \\ -\sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} \sum x_i y_i \\ \sum y_i \end{bmatrix}$$

$$= \frac{1}{N\sum x_i^2 - (\sum x_i)^2} \begin{bmatrix} N \sum x_i y_i - \sum x_i \sum y_j \\ \sum x_i^2 \sum y_i - \sum x_i \sum x_j y_j \end{bmatrix}$$

Example

- Best linear fit to (0,1), (1, 3), (2,4)
  - Calculate:
    - $N = 3$
    - $\Sigma_i x_i = 3$
    - $\Sigma_i y_i = 8$
    - $\Sigma_i x_i^2 = 5$
    - $\Sigma_i x_i y_i = 11$
  - $m = 9/6$, $b = 7/6$



6
5
4
3
2
1
y
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
x
x
x
x

Other best fits

- Best quadratic, cubic, etc.
- Best combination of sines and cosines (*"Fourier series"*).
- Other functions representing your preconceptions about the data.

The Lagrange condition

- Assuming f and C are “smooth,” at a boundary maximum point $\mathbf{x}_0$ where $\nabla g(\mathbf{x}_0) \neq \mathbf{0}$,

$$\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0)$$

for some scalar value λ.

What if we have more than one constraint?

Example

- The intersection of two planes, such as

  $$x + 2y - 3z = 6$$

  and

  $$x + y + z = 1$$

  is a line. What is the closest point on the line to the origin?

Lagrange with two constraints

- Assuming f is "smooth," and constrained by two "smooth" functions,
  - $g(\mathbf{x}) = 0$, and
  - $h(\mathbf{x}) = 0$.
- Lagrange's condition for a doubly constrained critical point is

$$\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0) + \mu \nabla h(\mathbf{x}_0)$$

for some scalar values λ and μ.

Example

- Objective function: $f(x,y,z) = x^2+y^2+z^2$
- Lagrange says:

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} =$$

$$\lambda\,(1\mathbf{i}+2\mathbf{j}-3\mathbf{k}) + \mu\,(1\mathbf{i}+1\mathbf{j}+1\mathbf{k})$$

Five unknowns (x,y,z,λ,μ). This vector equation represents three scalar eqns. We need two more…

Lagrange conditions

```
In[2]:= Solve[{2 x == lambda + mu, 2 y == 2 lambda + mu,
          2 z == 3 lambda + mu, x + 2 y - 3 z == 6, x + y + z == 1},
         {x, y, z, lambda, mu}]
```

$$\text{Out[2]=}\ \left\{\left\{x \to \frac{11}{6},\ y \to \frac{1}{3},\ z \to -\frac{7}{6},\ \text{lambda} \to -3,\ \text{mu} \to \frac{20}{3}\right\}\right\}$$

constraints

Approximating with differentials

- The differential way of writing gradients:

$$df = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z$$

or, as it is often written in science and engineering,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

Approximating with differentials

- Another point of view:

$$df = \nabla f \cdot d\mathbf{x}$$

(just like df = f´ dx)

Our book writes **h** in place of d**x**.

Approximating with differentials

- Example. Suppose you have a rectangle intended to have dimensions 3 by 4 cm.
- Estimate the change in the area if x is too big by 0.1 and y is too small by 0.2.
- If $A(x,y) = xy$, $dA = y\,\Delta x + x\,\Delta y$. With $x = 3$, $y = 4$, $\Delta x = 0.1$, $\Delta y = -0.2$,

$$dA = 0.4 - 0.6 = -0.2$$

The area is less by approximately 0.2 cm^2.

The End

of first lecture

Part Two, or

ΣΣΣΣΣΣΣΣΣΣΣΣΣΣ,

Δ Δ Δ Δ Δ Δ Δ Δ Δ Δ Δ Δ Δ Δ Δ Δ,

and

∫∫∫


COMPAQ

Each slice has width

Δx

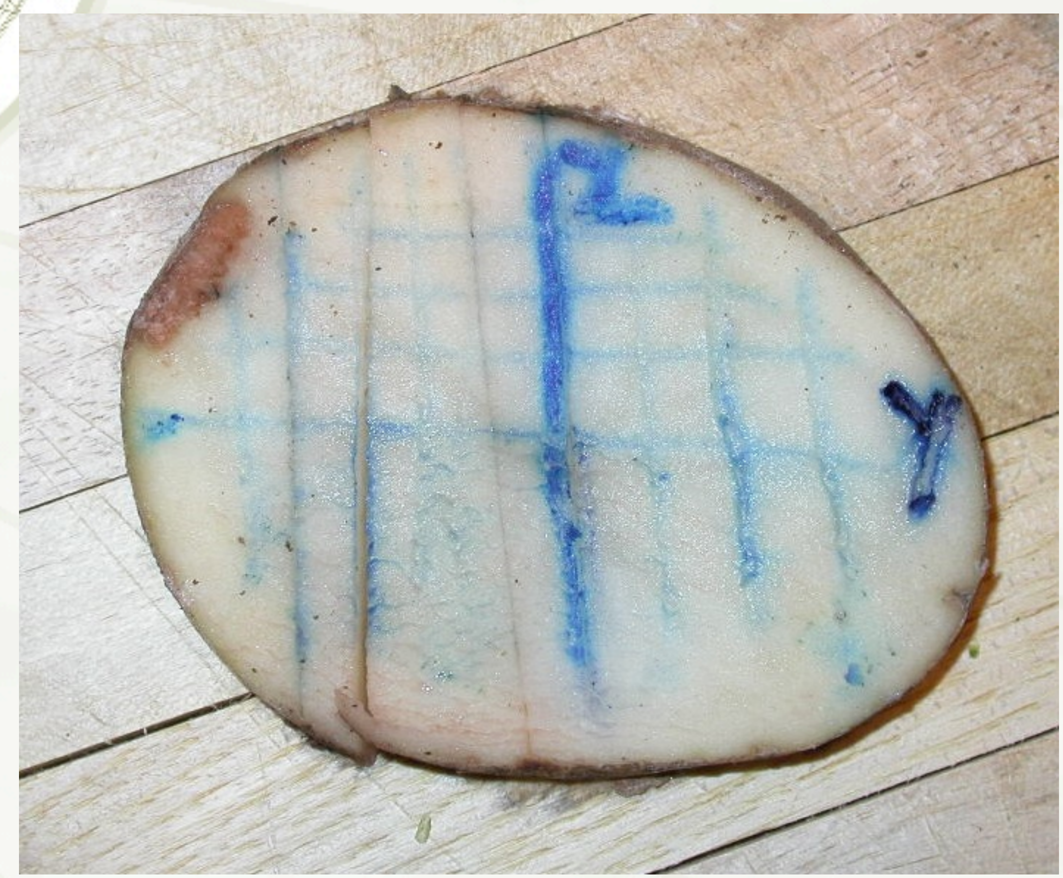
Each slice has volume

$A(x)\Delta x$

Total volume ≈ a Riemann sum

$\sum_i A(x_i)\Delta x$

Now take a close look at the slices:



Why, there’s a little coordinate grid on there! And, and…

Now take a close look at the slices:

So each A(x) is itself an integral of the form ∫z dy.

Suppose the height of a rectangular region is z = h(x,y).
The volume is an *iterated integral*,

Integrate over h(x,y) to get A(x).

$$Vol(\Omega) = \int_a^b \left(\int_c^d h(x,y)dy \right) dx$$

This potato has been sliced along the x-axis with width Δx.

What happens when we slice it a second time along the y-axis with width Δy?

FRIES!

The volume of a vertical French fry is (hgt • Δx Δy), (or pretty close)

Double Riemann whammy:

$$Vol(\Omega) = \int_a^b \left(\int_c^d h(x,y)dy \right) dx$$

This is a definite integral plugged into a second definite integral. In other words, it is a delicate kind of limit of a Rieman sum plugged into another limit of a Rieman sum. *Ouch!*

$$\sum_{j=1}^{N} \left(\sum_{k=1}^{M} h(x_j, y_k) \Delta x \Delta y \right).$$

Anatomy of the notation

$$\int_a^b \left(\int_c^d h(x,y)dy \right) dx = \int_a^b \int_c^d h(x,y)dydx = \int_R h(x,y)dxdy$$

$$\sum_{j=1}^{N} \left(\sum_{k=1}^{M} a_{jk} \right) = \sum_{j=1}^{N} \sum_{k=1}^{M} a_{jk}$$

Do the inside operation first. (*Does it matter*?)

Double Riemann whammy:

$$Vol(\Omega) = \int_a^b \left(\int_c^d h(x,y)dy \right) dx$$

This is the one we write and calculate.

This is the one we think about and get the computer to calculate.

$$\sum_{j=1}^{N} \left(\sum_{k=1}^{M} h(x_j, y_k) \Delta x \Delta y \right).$$

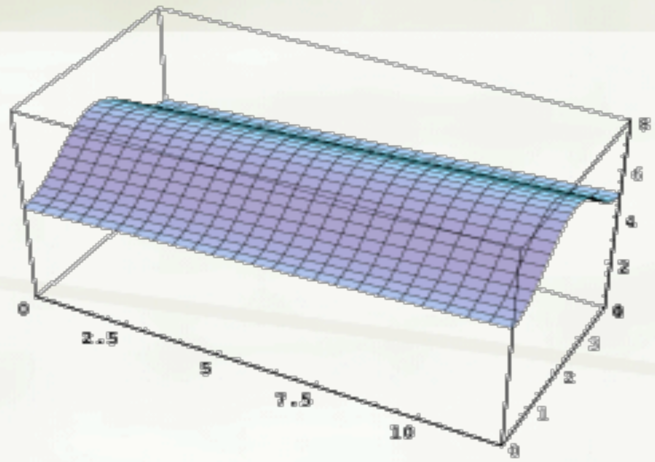
Examples:

A loaf of bread sits in a rectangular pan,

$R = \{0 \le x \le 12,\ 0 \le y \le 4\}$.

The height of the bread at point (x,y) is $5 + (x/12) - \cos(\pi y/2)$ (units are inches). What is the volume of the loaf?

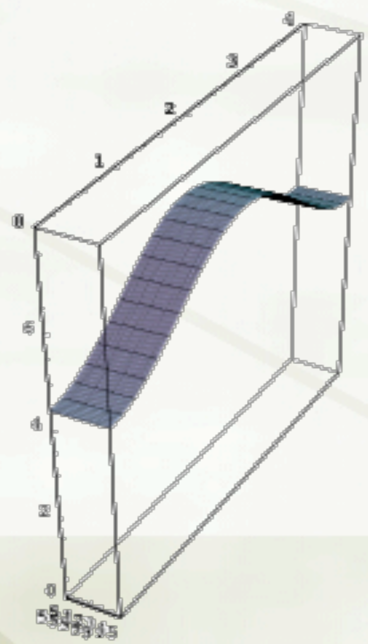
```
In[12]:=  Plot3D[5 + (x / 12) - Cos[Pi y / 2], {x, 0, 12},
            {y, 0, 4}, PlotRange → {0, 8}, BoxRatios → {2, 1, 1}]
```




```
Out[12]=  - SurfaceGraphics -
```

Here is one slice, of thickness Δx:

```
In[14]:=  Plot3D[5 + (x / 12) - Cos[Pi y / 2], {x, 5, 5.5},
            {y, 0, 4}, PlotRange → {0, 8}, BoxRatios → {1, 8, 8}]
```



First integral:

```
In[16]:= Integrate[5 + (x / 12) - Cos[Pi y / 2], {y, 0, 4}]
```

Out[16]= $\frac{60 + x}{3}$

Second integral:

```
In[17]:= Integrate[(60 + x) / 3, {x, 0, 12}]
```

Out[17]= 264

- How about doing it the other way?

First integral:

In[18]:= `Integrate[5 + (x / 12) - Cos[Pi y / 2], {x, 0, 12}]`

Out[18]= $66 - 12\,\mathrm{Cos}\left[\frac{\pi y}{2}\right]$

Second integral:

In[19]:= `Integrate[66 - 12 Cos[π y / 2], {y, 0, 4}]`

Out[19]= 264