Slicing, dicing, and integrating

Some announcements

- Next test: Thursday the 23rd!

Next week!

Some announcements

Als letztes Beispiel betrachten wir nun eine Fläche mit einem parabolischen Punkt, der isoliert in einem sonst sattelförmig ge... Gebiet liegt (Abb. 213); es ist der S. 169 beschriebene Affensattel. Bei dieser Fläche haben offenbar diejenigen Punkte parallele ...en, die zu dem parabolischen Punkt diametral liegen. Einer geschlossenen doppelpunktfreien Kurve um diesen Punkt herum entspricht also auf der Kugel eine geschlossene Kurve, die das sphärische Bild des Punkts zweimal umläuft[1]. Ebenso kann man offenbar isolierte parabolische Punkte mit sattelförmiger Umgebung konstruieren, bei denen das sphärische Bild einen einmaligen Umlauf in einen drei- oder beliebig vielfachen verwandelt. Geht man dagegen von einem isolierten parabolischen

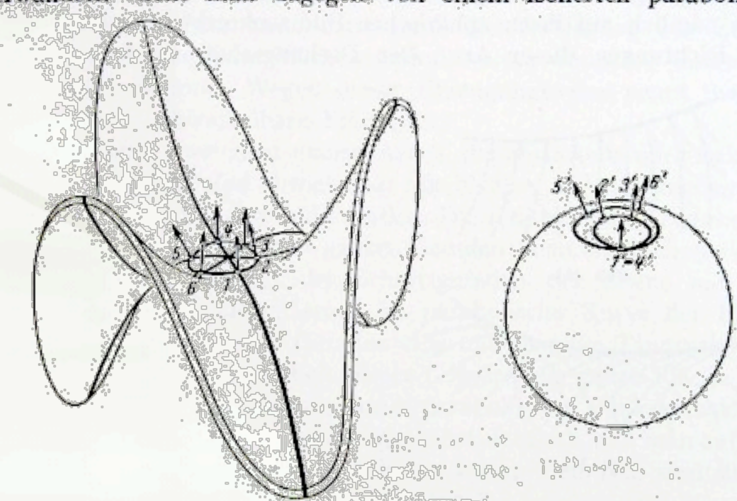


Abb. 213.

From D. Hilbert and S. Cohn-Vossen,
Anschauliche Geometrie (*Geometry and the Imagination*)

Some announcements

From D. Hilbert and S. Cohn-Vossen,
Anschauliche Geometrie (*Geometry and the Imagination*)


Congratulations to Jacob Schloss!
Das
Schloß

An interesting function called the monkey saddle can be expressed with the function $f(x,y)=x(x^2-3y^2)$ [1]. To find the interesting points of this function, such as the stationary points and the general shape, it is likely to be useful to take the gradient and evaluate some sample normal vectors. The gradient is calculated to be $\nabla f(x,y)=[3x^2-3y^2]\hat{i}+[-6yx]\hat{j}$. This shows there is a stationary point at (0,0,0), and nowhere else. To find if the center point is a saddle, the Hessian matrix can be created and applied.

Hessian Matrix: $H(x,y)=\begin{bmatrix} 6x & -6y \\ -6y & -6 \end{bmatrix} \rightarrow D(x,y)=-36x-36y^2$

The determinant D(0,0) = 0, so this cannot be used to find the properties of the origin. Instead some representative normal vectors can be picked and examined. The normal vector is $\vec{N}=\nabla f(x,y,z)=[-3x^2+3y^2]\hat{i}+[6yx]\hat{j}+\hat{k}$. A table of values follows.

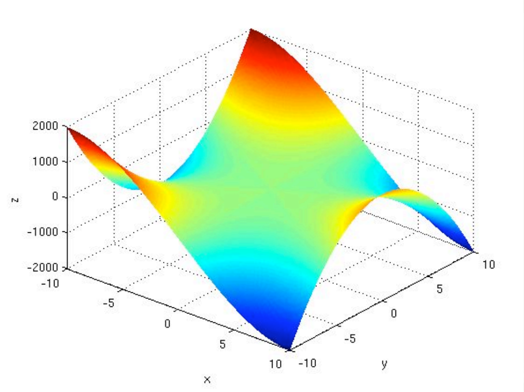
Point	$\nabla f(x,y,z)$	Tangent plane $(r-r_0)\cdot\vec{N}=0$
(0, 0, 0)	$\hat{k}$	$\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}-\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)\cdot\vec{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}=z=0$
(0, 1, 0)	$3\hat{i}+\hat{k}$	$\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}-\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)\cdot\vec{\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}=3(x)+(z)=0$
(1, 1, -2)	$6\hat{j}+\hat{k}$	$\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}-\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right)\cdot\vec{\begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}}=6(y-1)+(z-2)=0$
(1, 0, 1)	$-3\hat{i}+\hat{k}$	$\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}-\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)\cdot\vec{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}=-3(x-1)+(z-1)=0$
(1, -1, -2)	$-6\hat{j}+\hat{k}$	$\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}-\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}\right)\cdot\vec{\begin{bmatrix} 0 \\ -6 \\ 1 \end{bmatrix}}=-6(y+1)+(z+2)=0$
(0, -1, 0,)	$3\hat{i}+\hat{k}$	$\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}-\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}\right)\cdot\vec{\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}=3(x)+(z)=0$
(-1, -1, 2)	$6\hat{j}+\hat{k}$	$\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}-\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}\right)\cdot\vec{\begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}}=6(x+1)+(z-2)=0$
(-1, 0, -1)	$-3\hat{i}+\hat{k}$	$\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}-\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}\right)\cdot\vec{\begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}}=-3(y)+(z+1)=0$
(-1, 1, 2)	$-6\hat{j}+\hat{k}$	$\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}-\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}\right)\cdot\vec{\begin{bmatrix} 0 \\ -6 \\ 1 \end{bmatrix}}=-6(y-1)+(z-2)=0$

The tangent plane of z = 0 and normal vector of $\hat{k}$ at the point (0,0,0) also prove it is a stationary point. Moving up along the y axis, the tangent plane at (0,1,0) slopes downwards in the positive x direction as the x component of the normal is positive. Looking at the normal vectors for the points (1,0,1) and (1,1,-2), the y component of the normal increases as one moves from and to the respective points, indicating the function's slope is increasing in magnitude in the first quadrant as x and y increase. The function is moving down hill because the z value of the function at the point (1,1,-2) is lower than that of the origin, (1,0,1), and (0,1,0). Likewise for the forth quadrant, the normal at (1,-1,-2) also has a greater y component than the normal at (1,0,1), indicating the function is also getting steeper as y decreases and x increases from the origin. Again, this movement is downhill because the z value of the function is lower at (1,-1,-2) than at the origin, (1,0,1), and (0,-1,0).

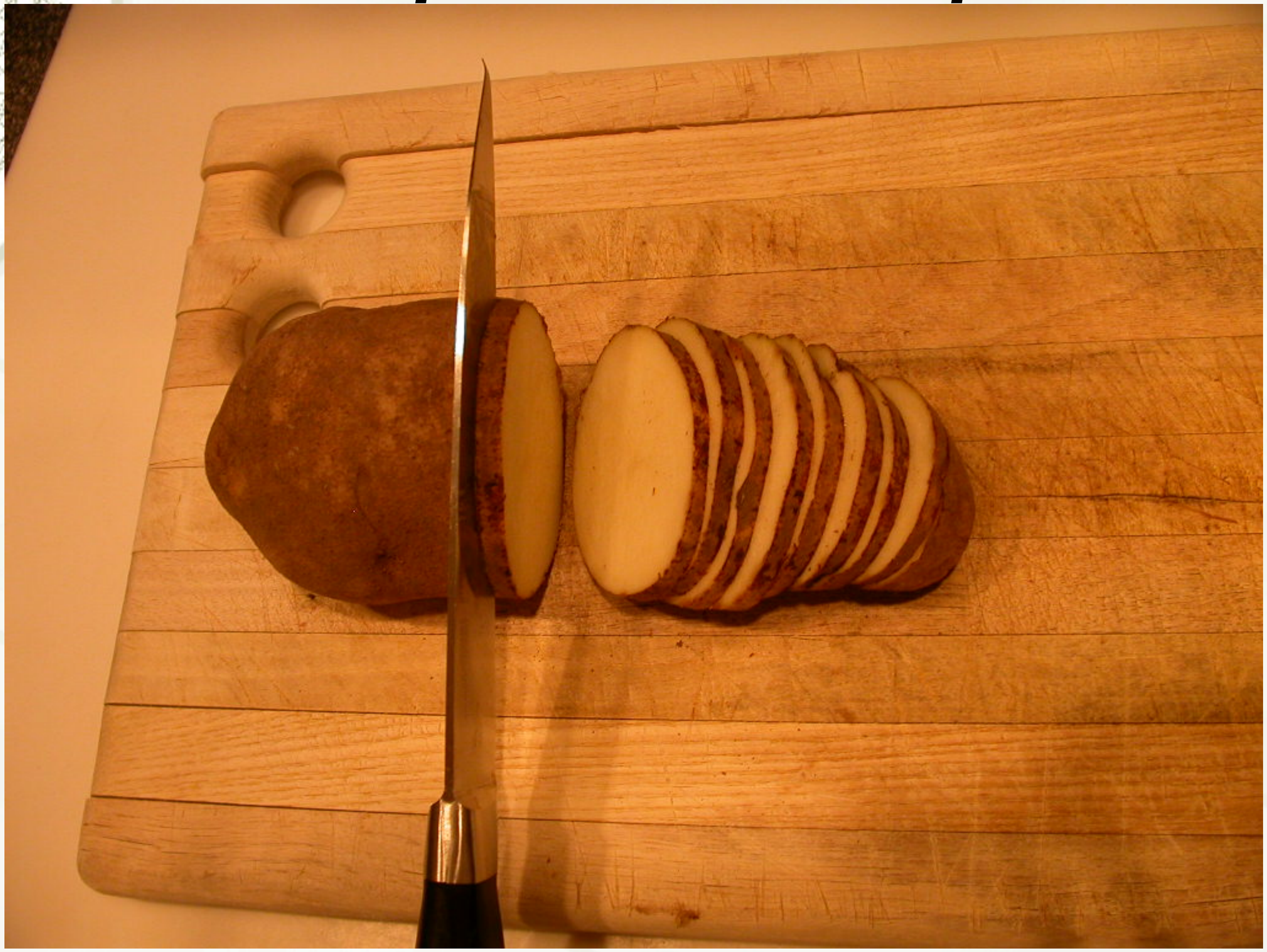
Moving to the second quadrant, and comparing the normals for (x,y) = (-1,0), (-1,1), and (0,1), we can see that the function increases in the direction of $-1\hat{i}+\hat{j}$ from the origin, and decreases to either side as the function is valued higher at (-1,1) and the normal swings from downhill along decreasing y, $\vec{N}=-6\hat{j}+\hat{k}$, at (x,y) = (-1,1) to flat in y; $-3\hat{i}+\hat{k}$ at (-1,0) and $3\hat{i}+\hat{k}$ at (0,1). The forth quadrant similarly has a peak at (-1,-1) and slopes back in a fashion symmetric to the third quadrant.

As the function moves from the origin to (-1,0), the normal shifts form vertical to pointing somewhat towards the negative x direction and the value of the function decreases, indicating that the function moves downslope along the x axis between the peaks at x,y = (-1,1) and (-1,-1). Conversely, as the function moves in the positive x direction, the normal continues to point towards the negative x direction, and the function value increases, so the function is moving uphill and a peak is centered somewhere along the positive x axis.

Therefore, the monkey saddle would have the saddle region centered on (0,0,0), with the two leg depressions in the direction of $\hat{i}+\hat{j}$ and $\hat{i}-\hat{j}$ from the origin, the tail depression is centered along the negative x axis and the saddle horn is centered on the positive x axis.



In our previous episode…



How do you slice it?

In our previous episode…

Or dice it?

FRIES!

The volume of a vertical French fry is (hgt • $\Delta x \Delta y$), (or pretty close)

How about some problems that are just like the HW?

How about some problems that are just like the HW?

1. OK – Evaluate the double sum

$$\sum_{i=1}^{3}\sum_{j=1}^{3} 2^{i-1}3^{j+1} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3} 2^{i-1}3^{j+1} = \underline{\hspace{10cm}}$$

How about some problems that are just like the HW?

1. OK – Evaluate the double sum…

2. Let $f(x,y) = x+2y$ on $R = \{0 \le x \le 2,\ 0 \le y \le 1\}$. And let us partition the region by $P = P_1 \times P_2$. where

   $P_1 = \{0,1,3/2,2\}$ and $P_2 = \{0,1/2,1\}$.

   Find $L_f(P)$ and $U_f(P)$.

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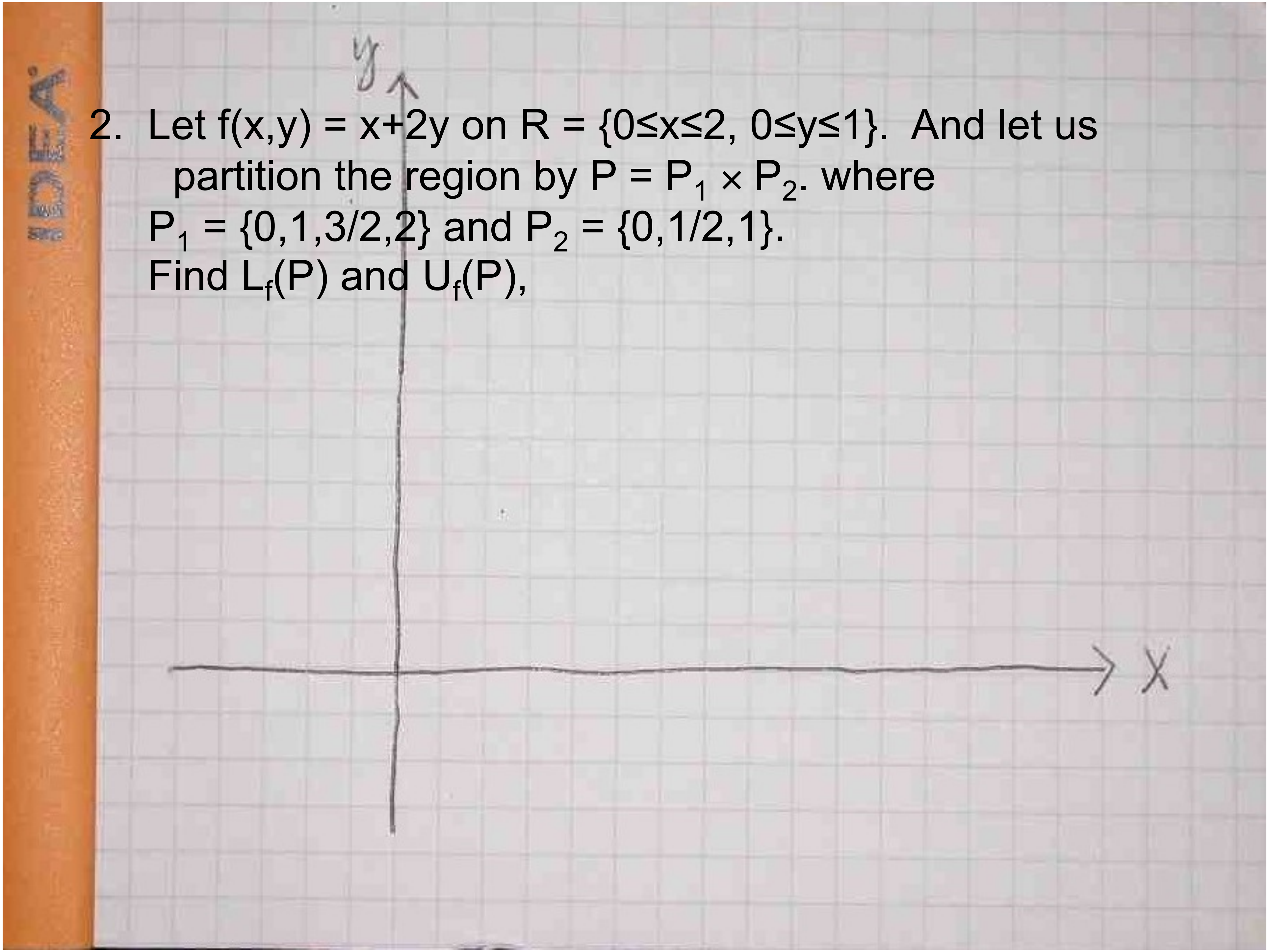
Find $L_f(P)$ and $U_f(P)$,



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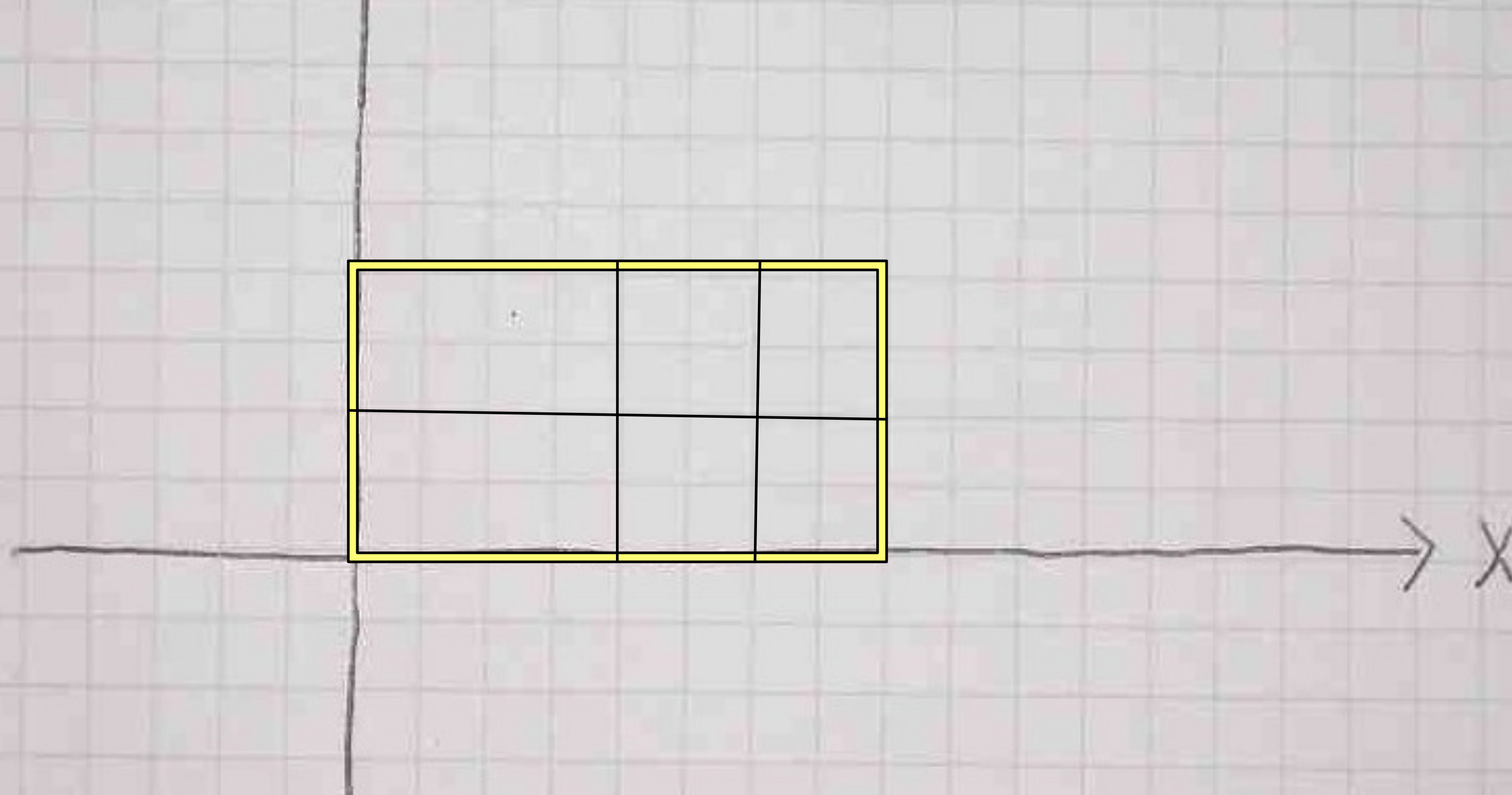

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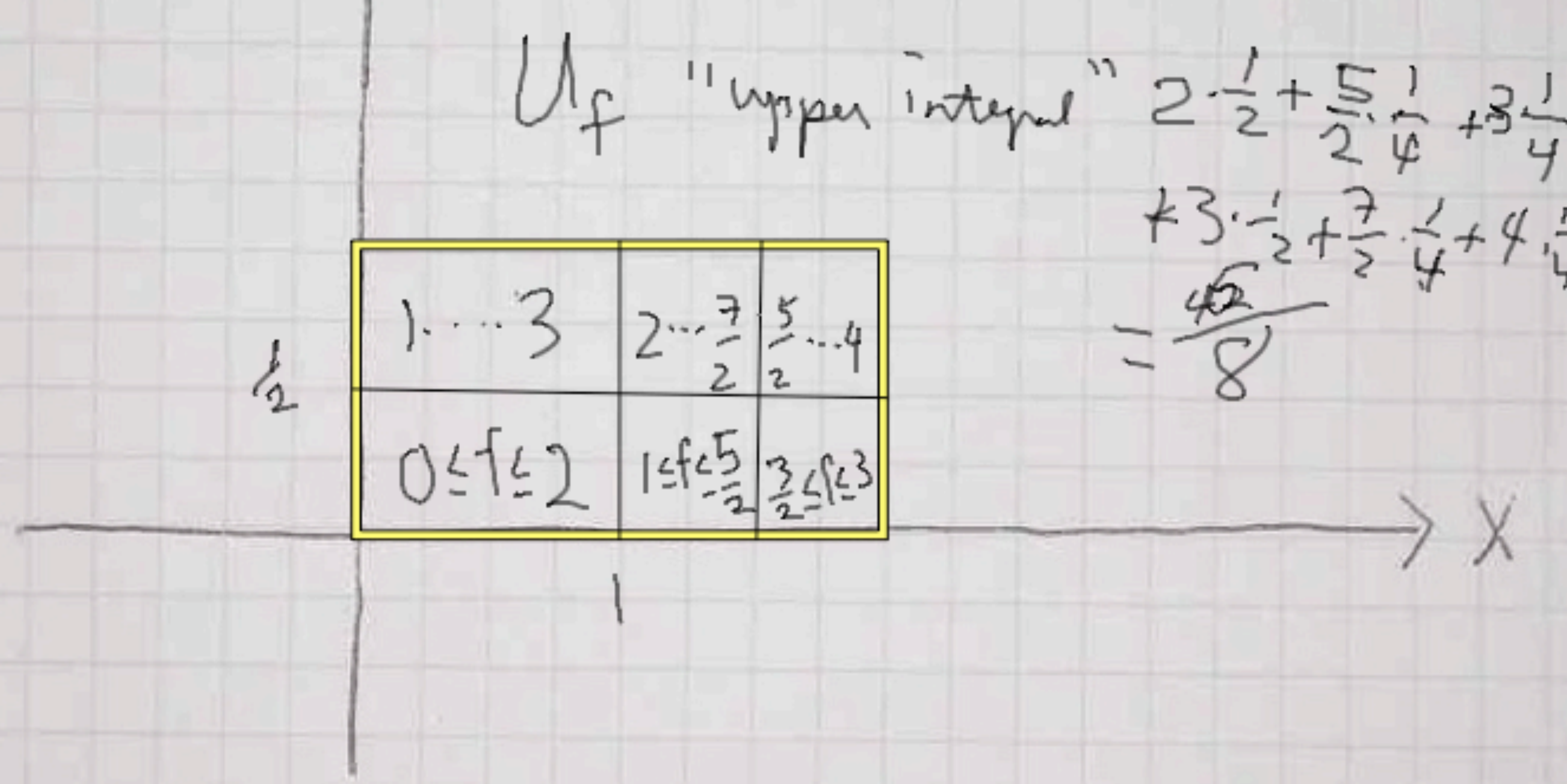
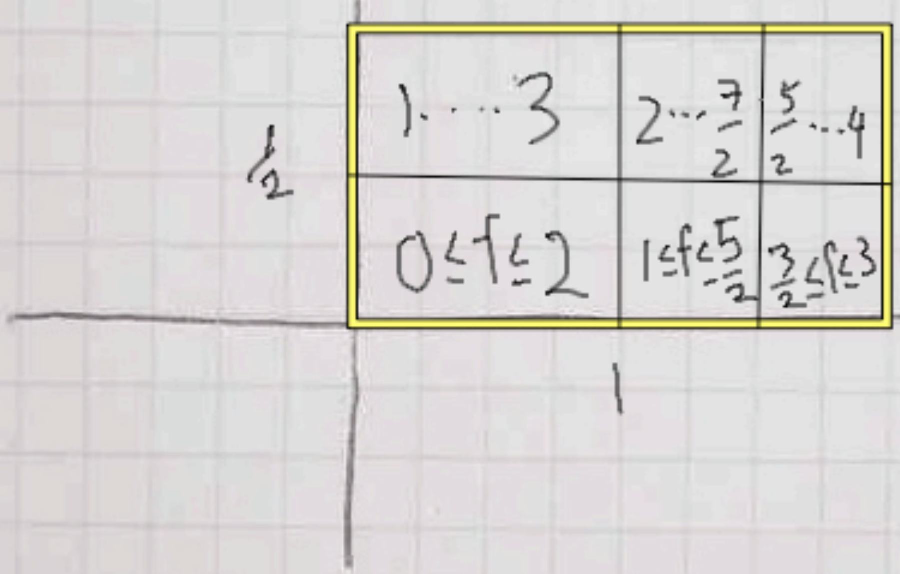


2. Let $f(x,y) = x+2y$ on $R = \{0 \le x \le 2, 0 \le y \le 1\}$. And let us partition the region by $P = P_1 \times P_2$. where $P_1 = \{0,1,3/2,2\}$ and $P_2 = \{0,1/2,1\}$. Find $L_f(P)$ and $U_f(P)$,

U_f "upper integral" $2 \cdot \frac{1}{2} + \frac{5}{2} \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} + 3 \cdot \frac{1}{2} + \frac{7}{2} \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} = \frac{45}{8}$



$U_L = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + \frac{3}{2} \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + \frac{5}{2} \cdot \frac{1}{4} = \frac{?}{8}$

Practicalities of doing double integrals.

A double integral

is

an iterated integral.

Practicalities of doing double integrals.

First integral:

```
In[16]:= Integrate[5 + (x / 12) - Cos[Pi y / 2], {y, 0, 4}]
```

Out[16]= $\frac{60 + x}{3}$

Second integral:

```
In[17]:= Integrate[(60 + x) / 3, {x, 0, 12}]
```

Out[17]= 264

- How about doing it the other way?

First integral:

In[18]:=
```
Integrate[5 + (x / 12) - Cos[Pi y / 2], {x, 0, 12}]
```

Out[18]= $66 - 12\,\mathrm{Cos}\left[\frac{\pi y}{2}\right]$

Second integral:

In[19]:=
```
Integrate[66 - 12 Cos[π y / 2], {y, 0, 4}]
```

Out[19]= 264

How about some problems that are just like the HW?

3. Evaluate

$$\int_R \int \sin(x)\sin(y)\,dx\,dy = \underline{\hspace{6cm}}$$

where $R = \{0 \leq x \leq \pi, 0 \leq y \leq \frac{\pi}{2}\}$.

$$\int\!\!\int_R \sin(x)\sin(y)\,dx\,dy = \rule{10cm}{0.4pt}$$

where $R = \{0 \leq x \leq \pi, 0 \leq y \leq \frac{\pi}{2}\}$.

What is the average value of sin(x) sin(y) on that rectangle?

It's time for …

Guess the theorem!

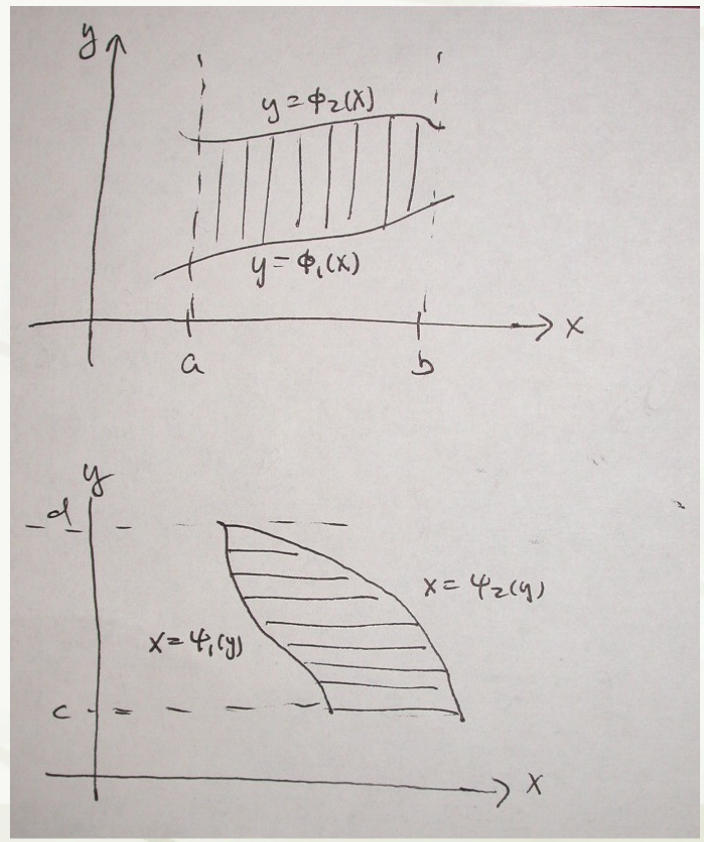
What is the average value of sin(x) sin(y) on that rectangle?

$$\frac{\iint_R f(x,y)\,dx\,dy}{\text{area}(R)} = \frac{2}{(\pi^2/2)} = \frac{4}{\pi^2}$$

What if the integration region is not a rectangle?

What if the integration region is not a rectangle?

- Easy cases:

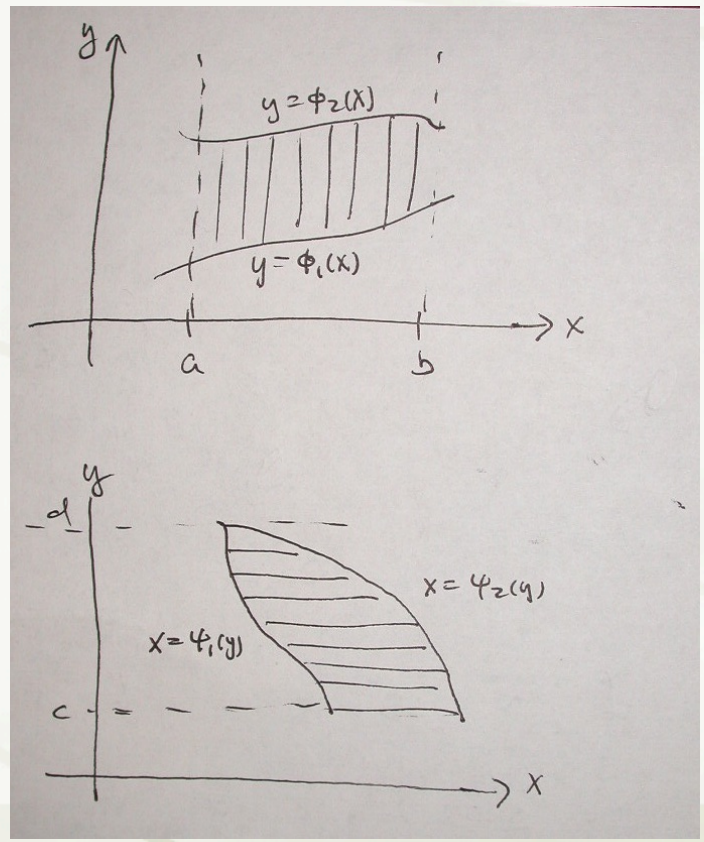


What if the integration region is not a rectangle?

- Easy cases:
  - Example:

    $0 \le x \le 2,$

    $x \le y \le 2$



What if the integration region is not a rectangle?

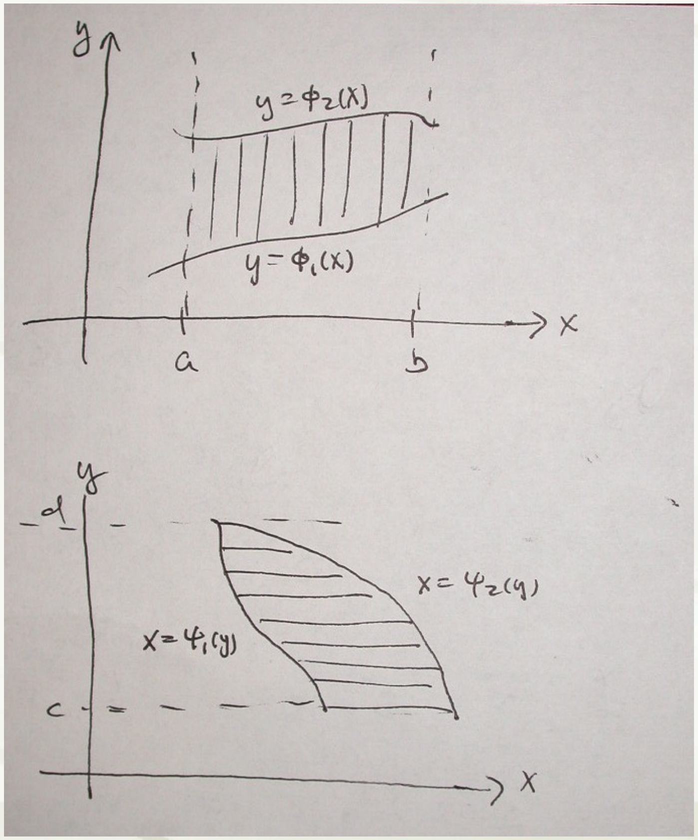
- Easy cases:
  - Example:

    $0 \le x \le 2,$

    $x \le y \le 2$
  - Example:

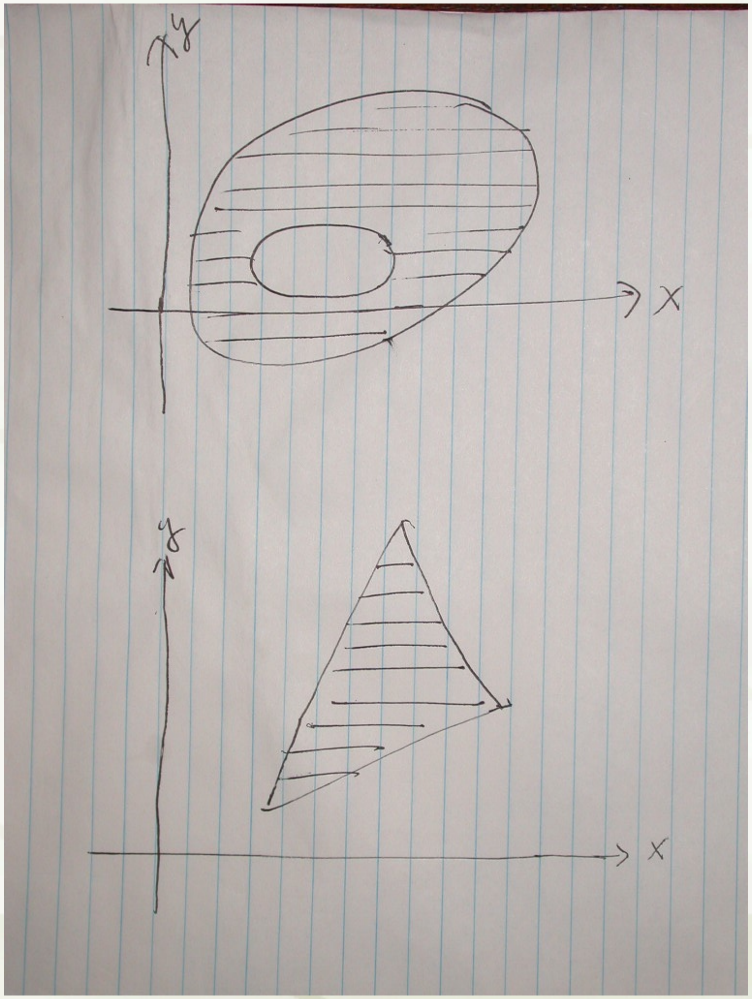
    $1-y \le x \le y$

    $?? \le y \le 4$



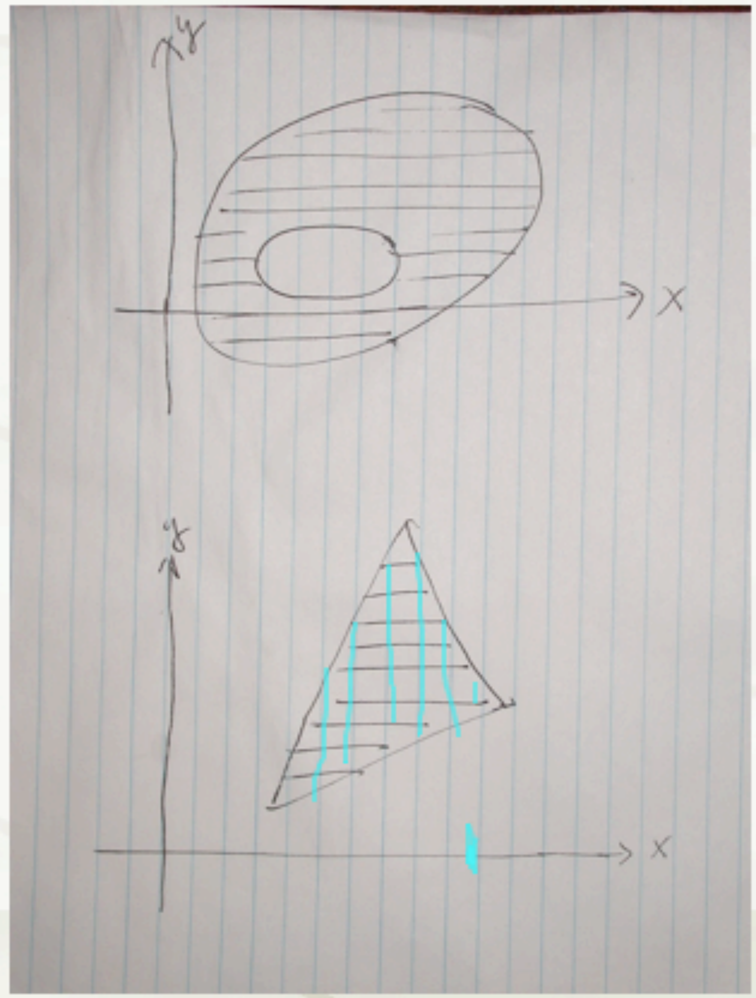
What if the integration region is not a rectangle?

- Not so easy cases:



What if the integration region is not a rectangle?

- Not so easy cases:

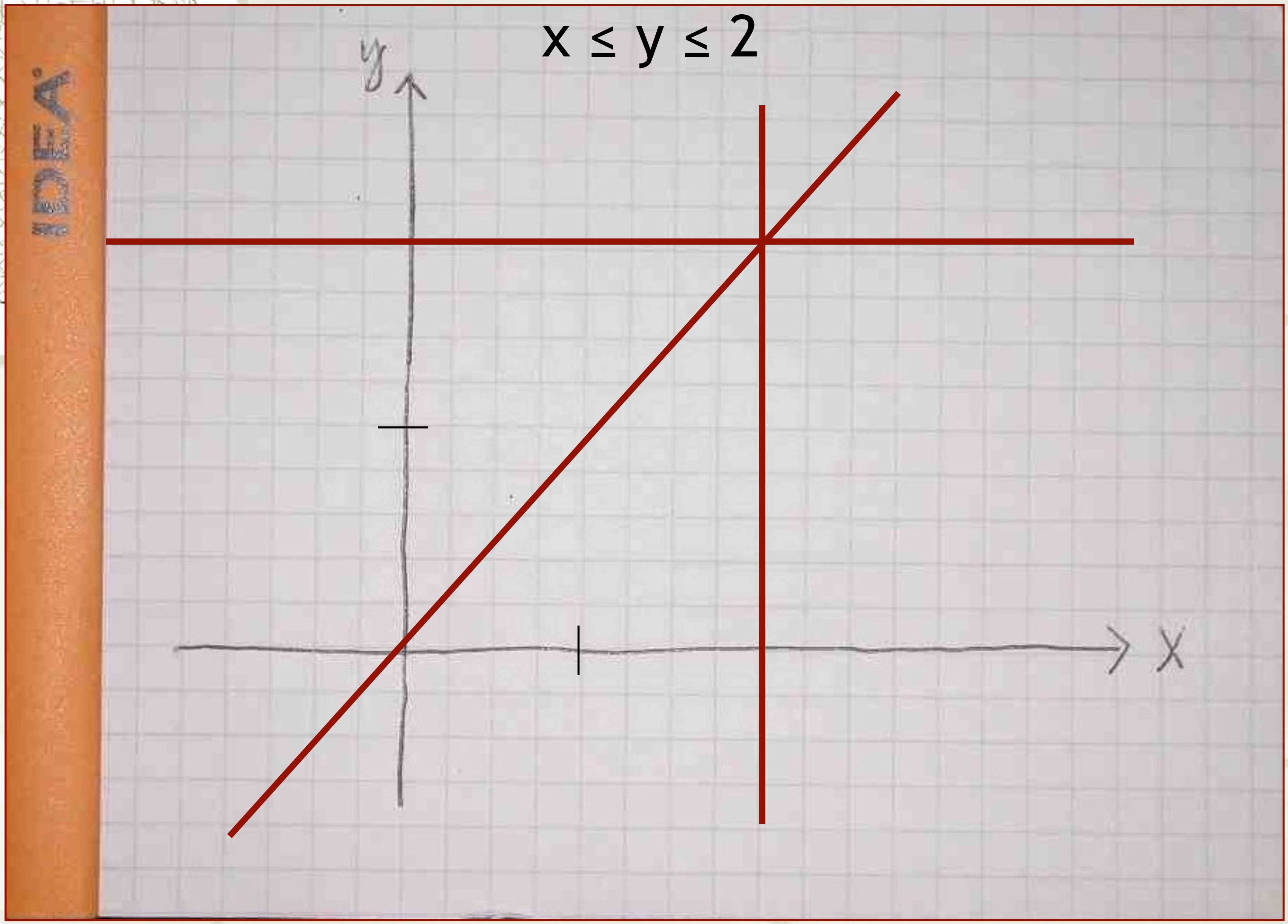


Another game:

- Let’s take one favorite function, like $f(x,y) = xy$, and integrate it over lots of regions.
  - What does the integral of xy over a region in the first quadrant ($x,y > 0$) represent?
  - What if the region is in the second quadrant ($x < 0$, $y > 0$)?

$0 \leq x \leq 2,$

$x \leq y \leq 2$



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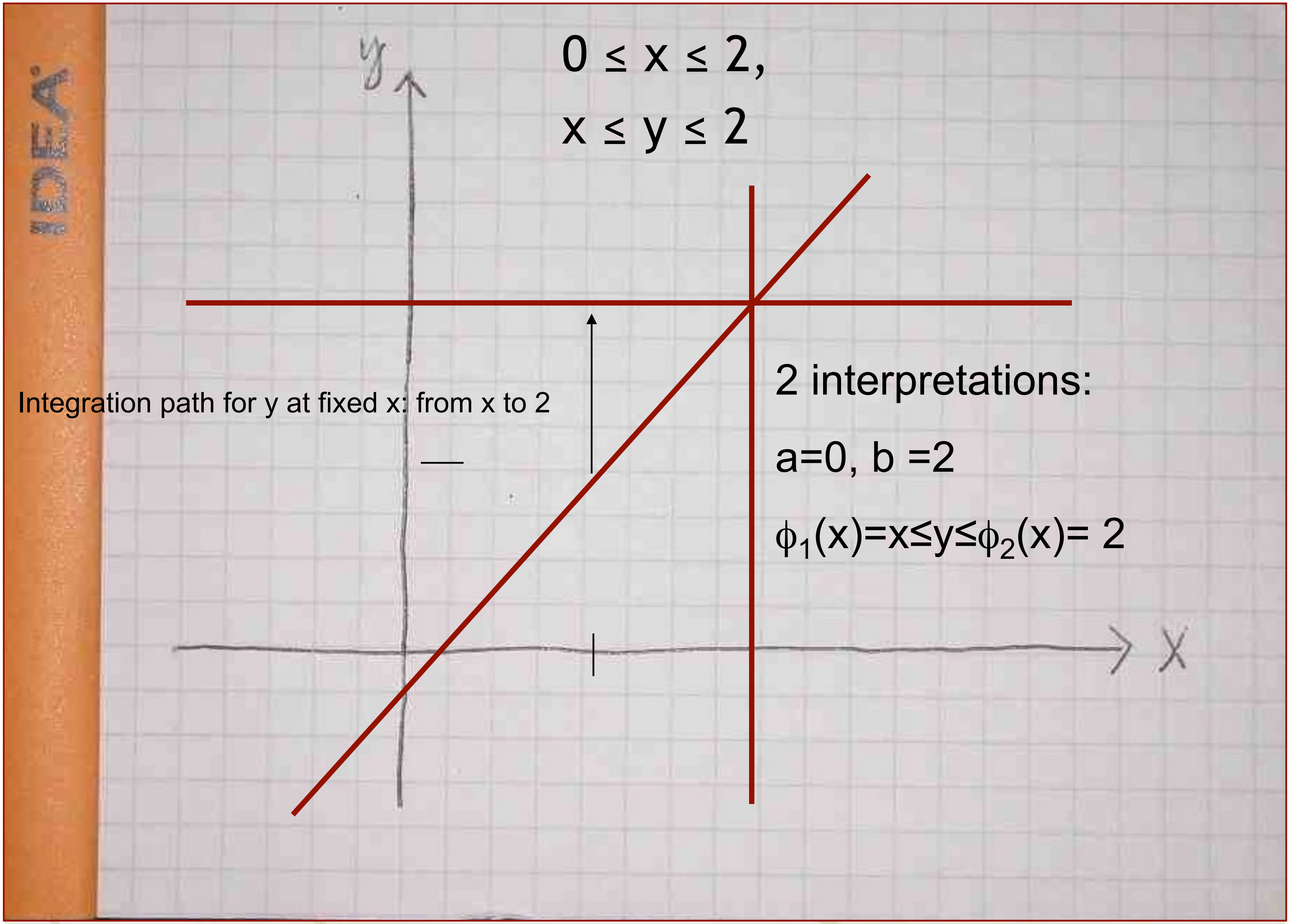


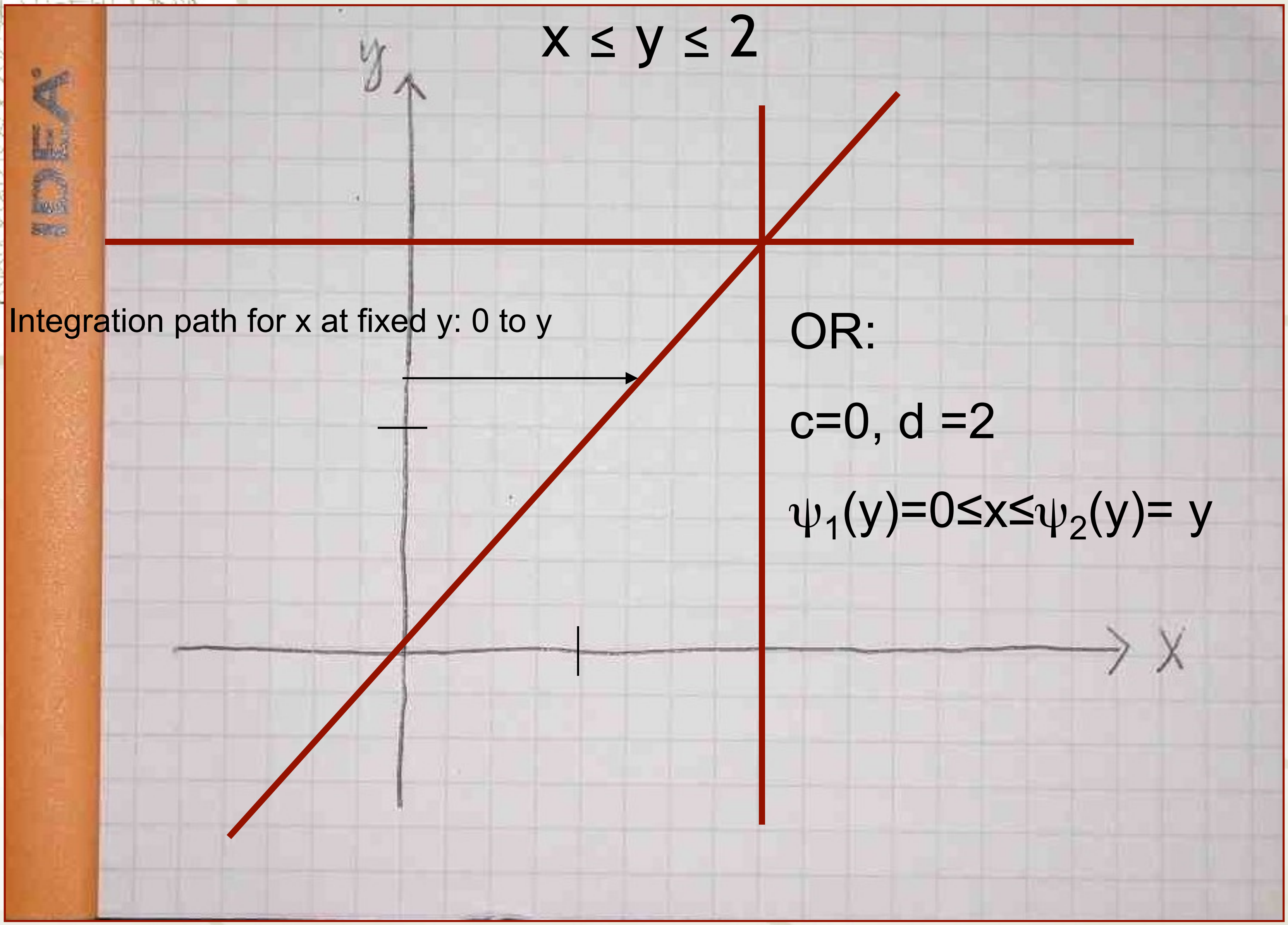
2 interpretations:

a=0, b =2

$\phi_1(x)=x\le y\le\phi_2(x)= 2$

$0 \le x \le 2,$

$x \le y \le 2$



Integration path for x at fixed y: 0 to y

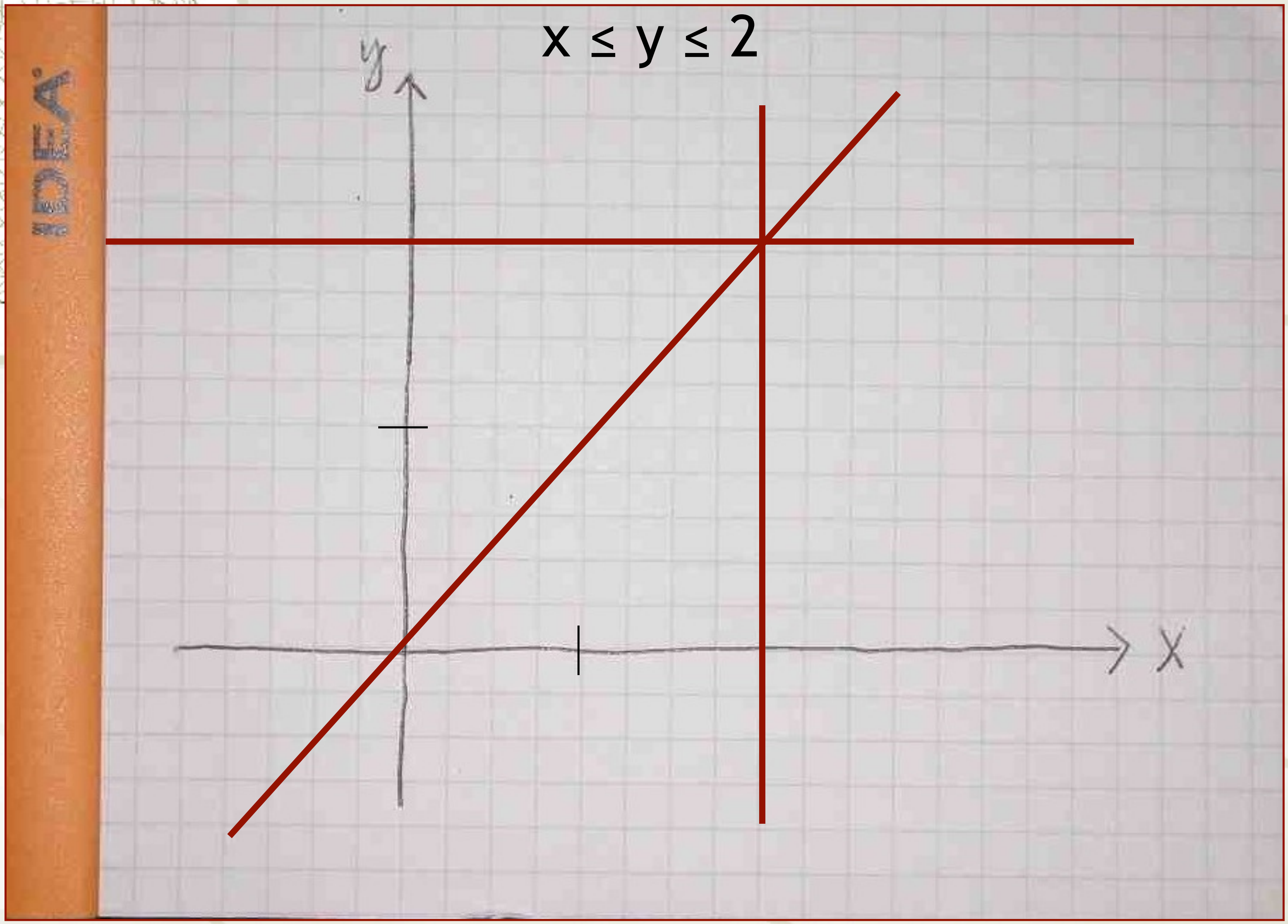
OR:

$c=0,\ d=2$

$\psi_1(y)=0 \le x \le \psi_2(y)= y$

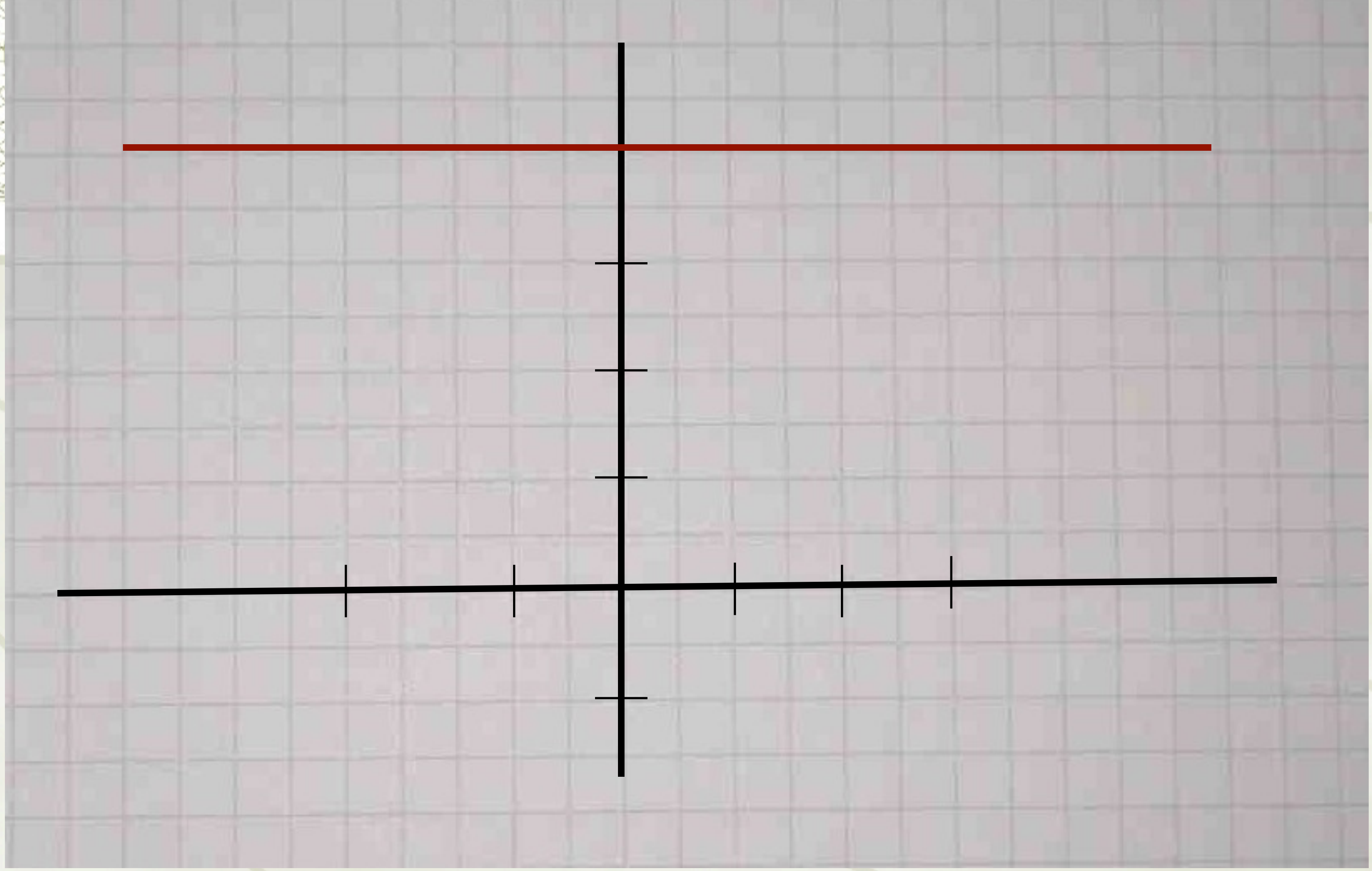
$0 \le x \le 2,$

$x \le y \le 2$



$1-y \leq x \leq y$

$?? \leq y \leq 4$



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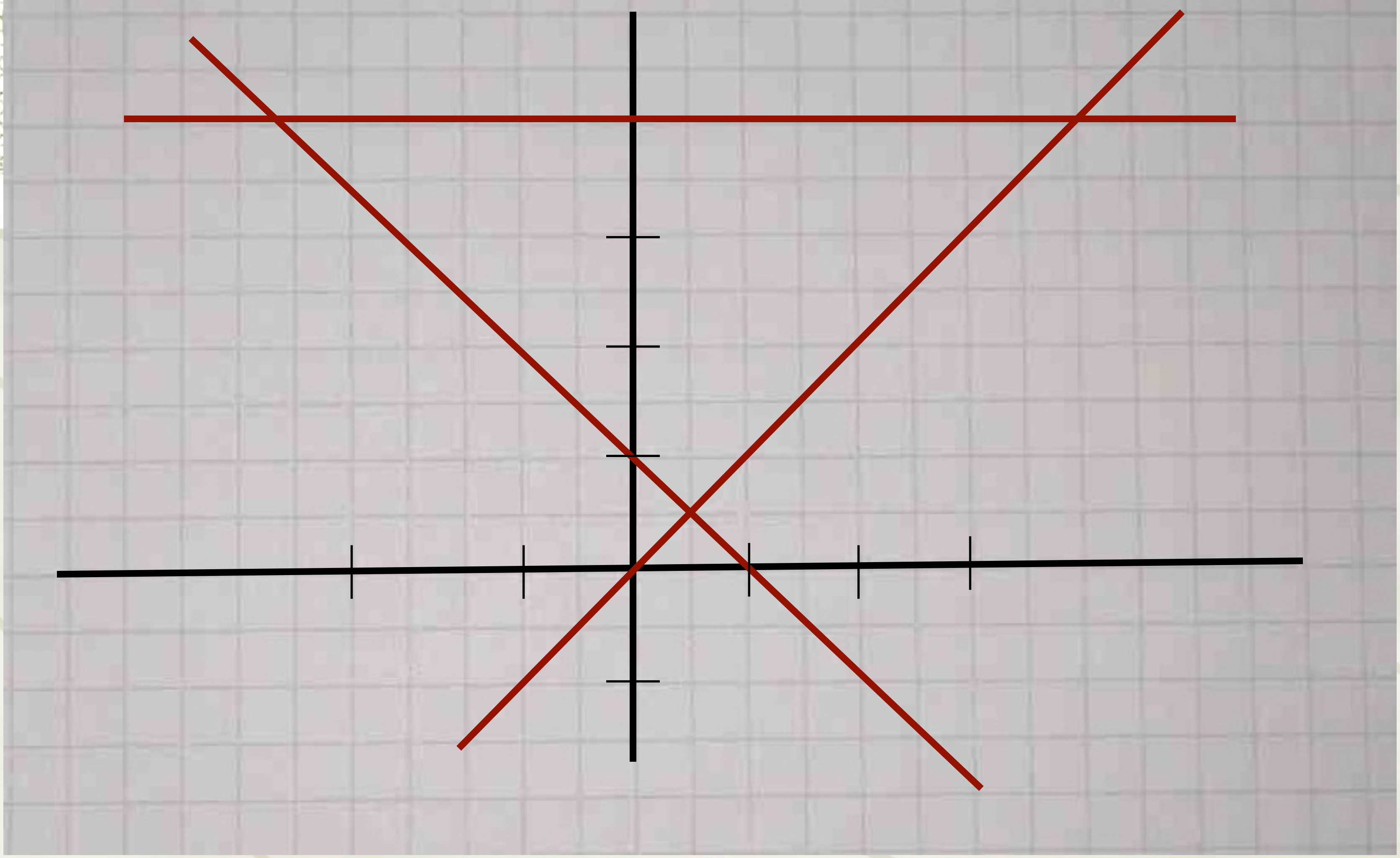

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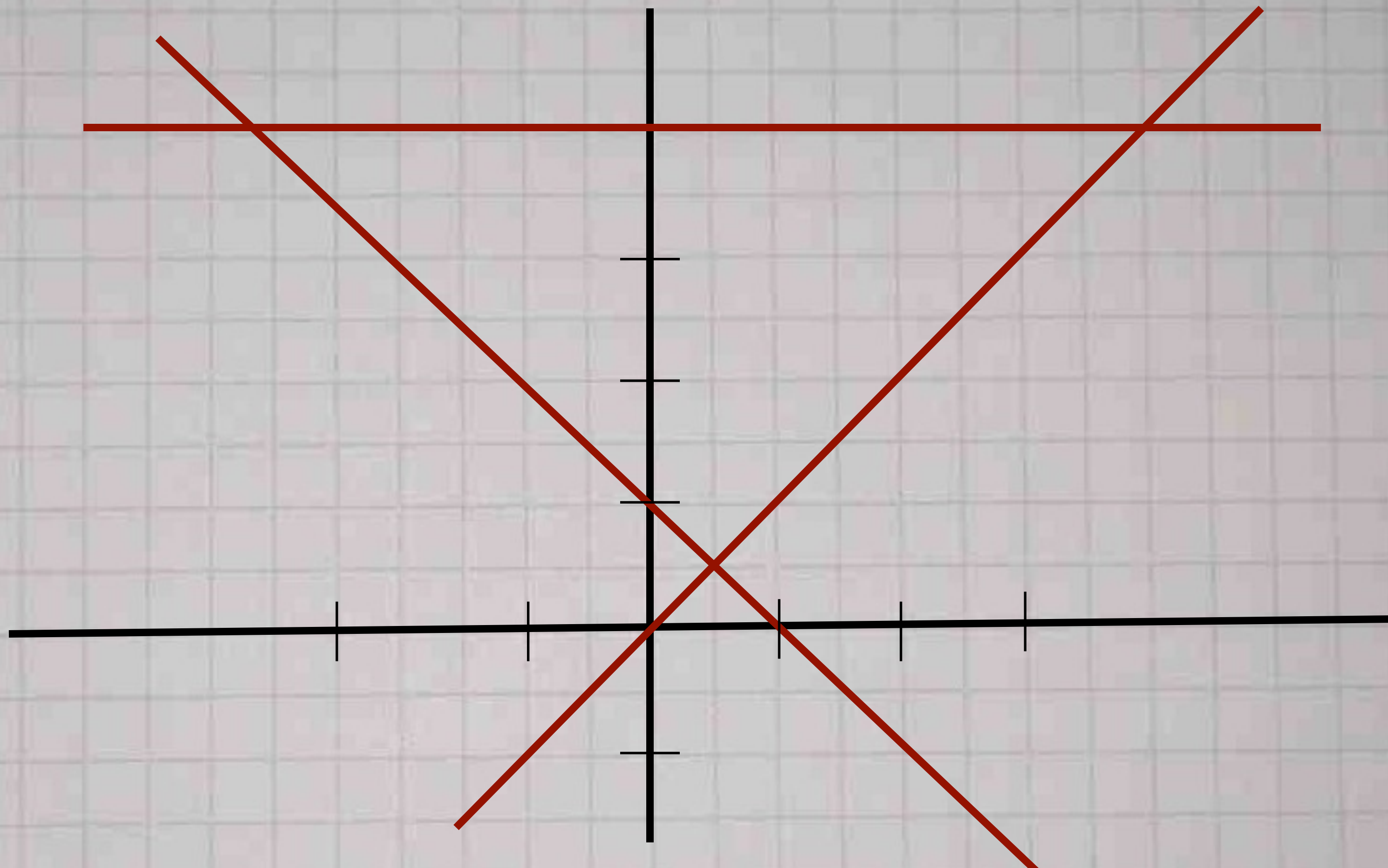

1-y ≤ x ≤ y

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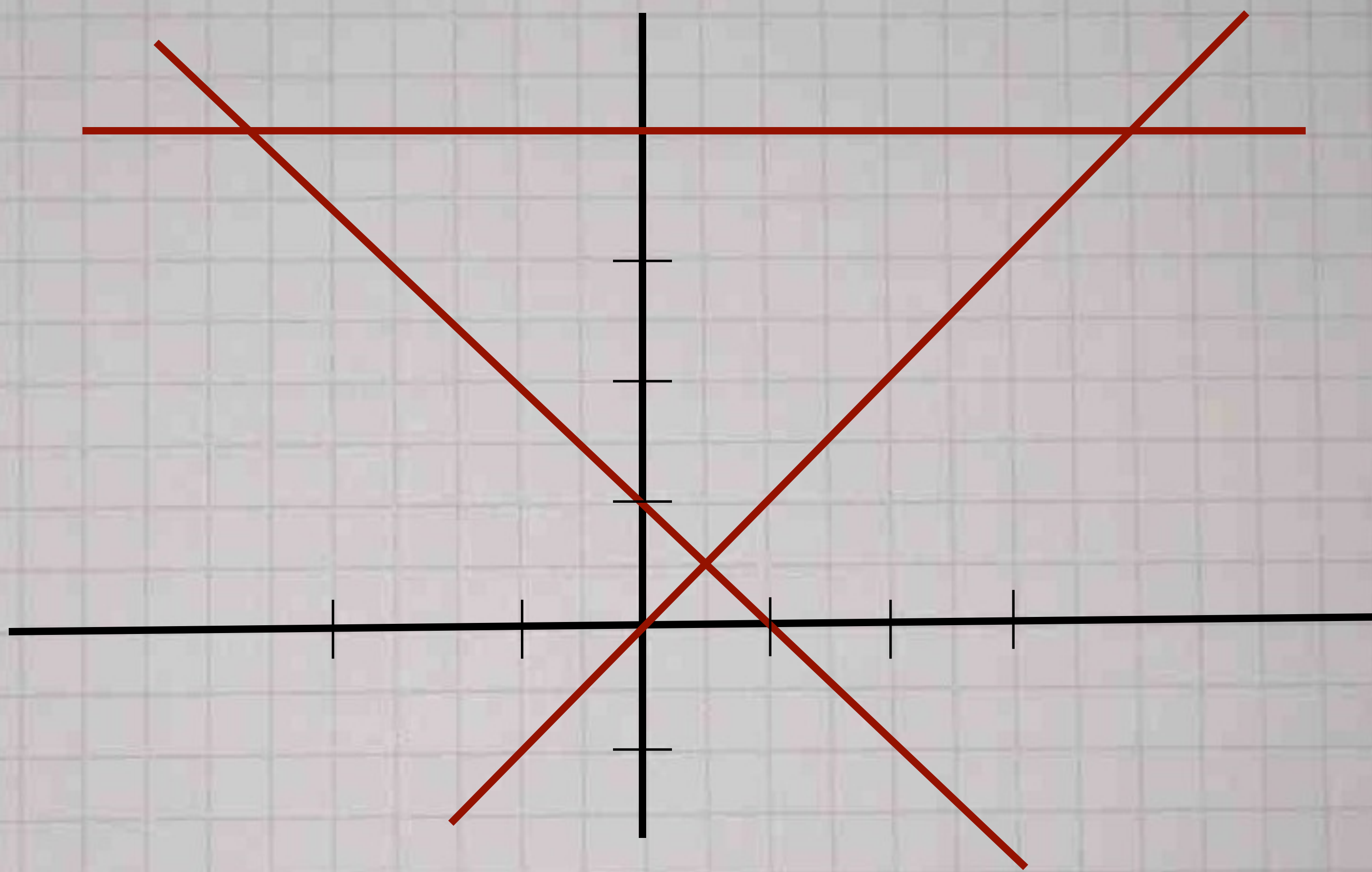
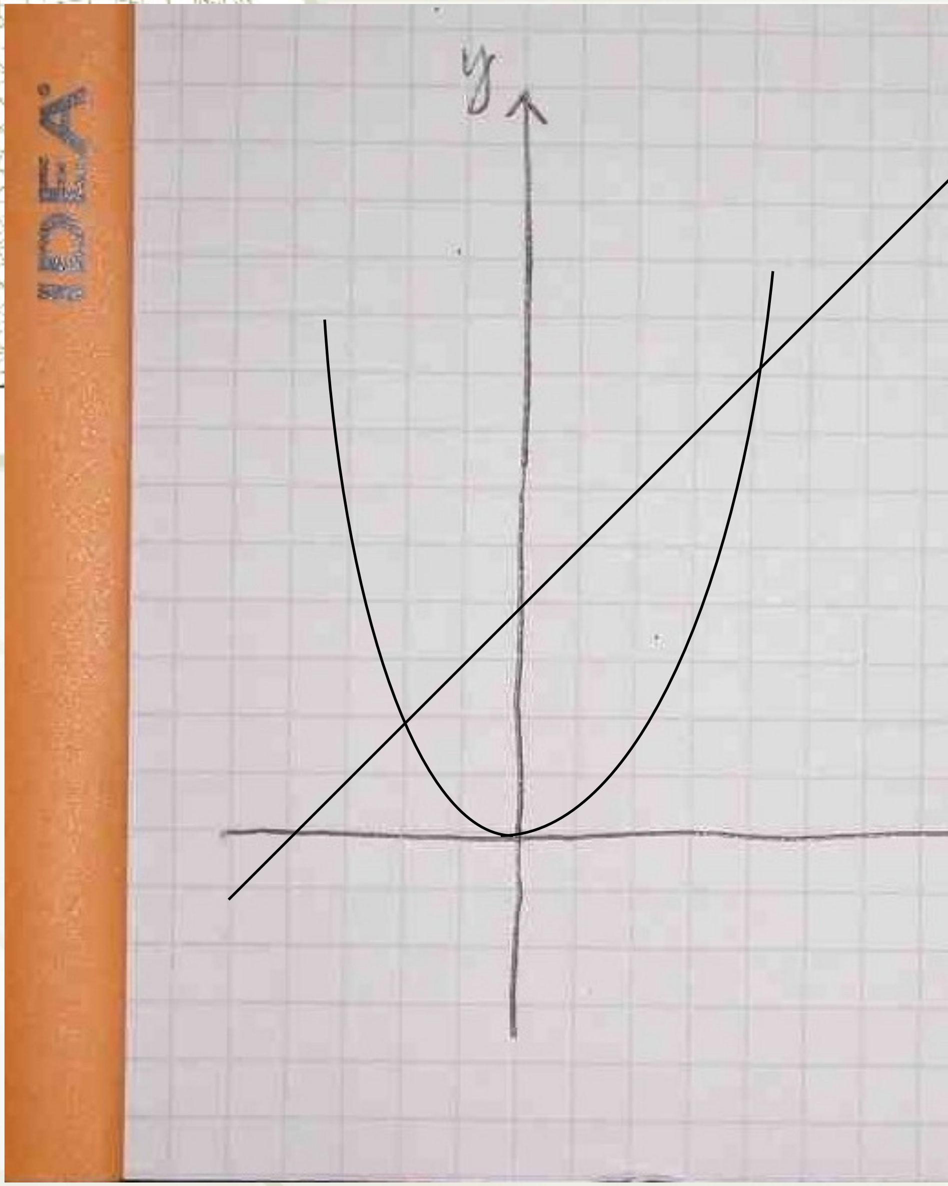
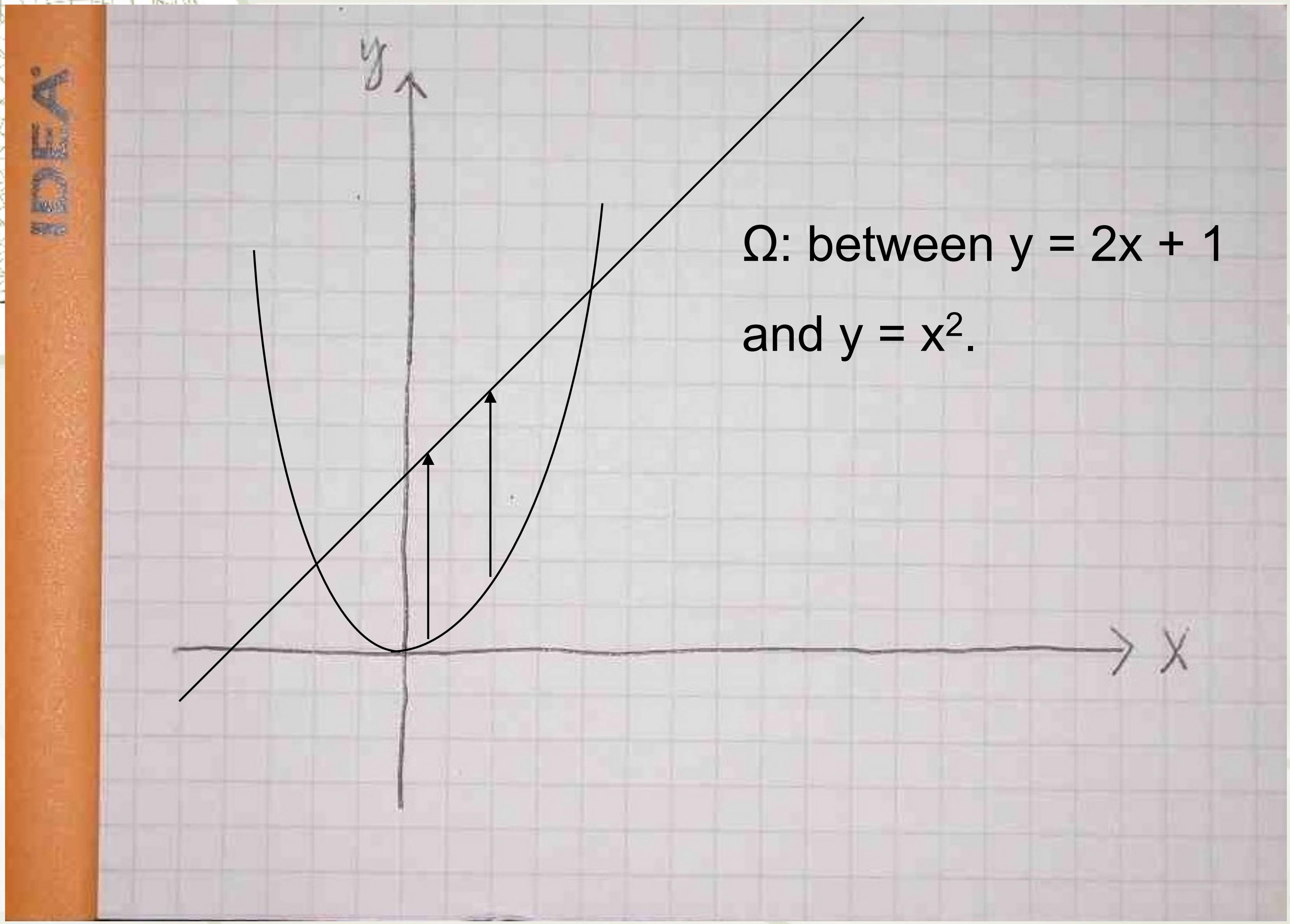


Example



Ω: between $y = 2x + 1$

and $y = x^2$.



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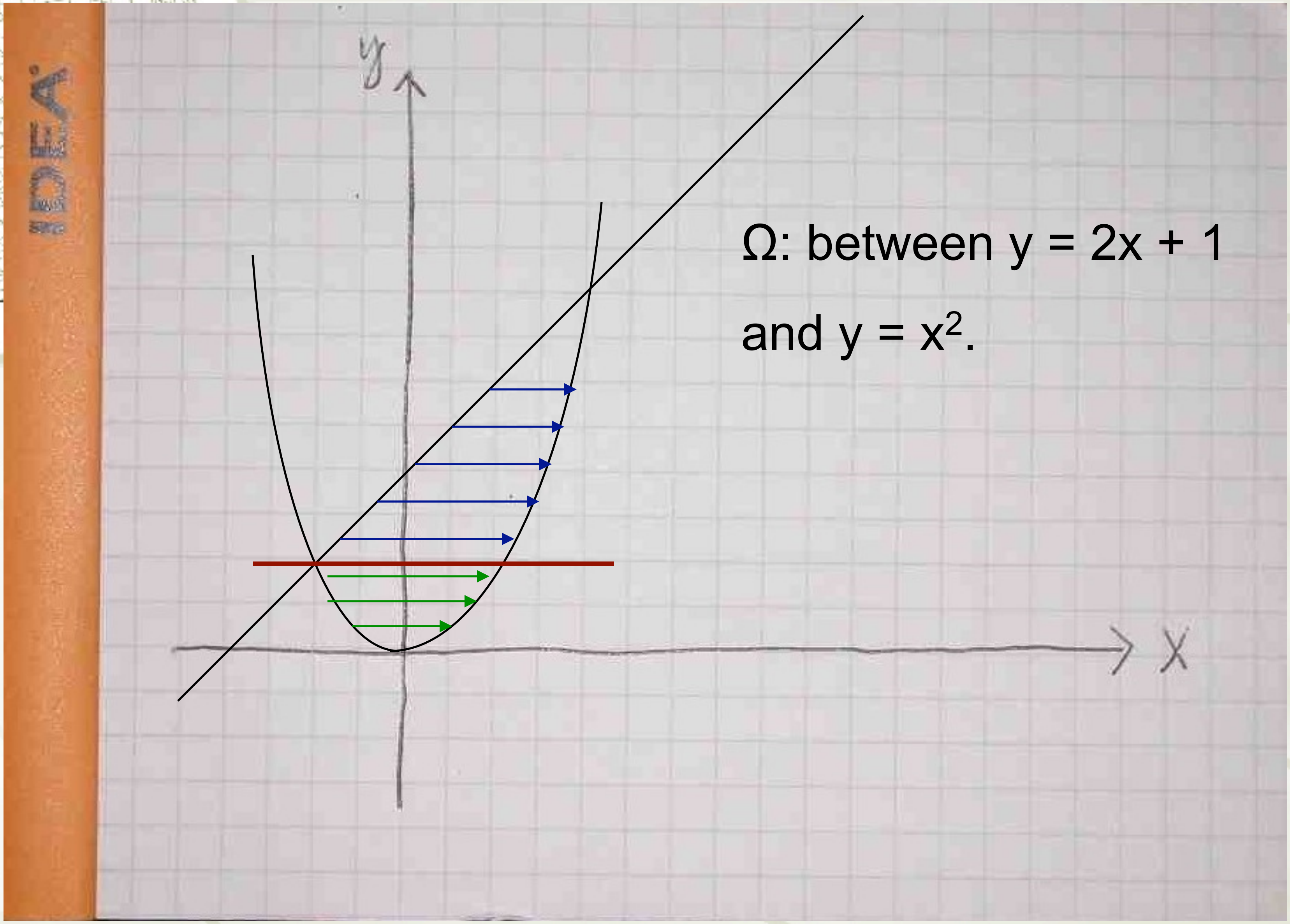

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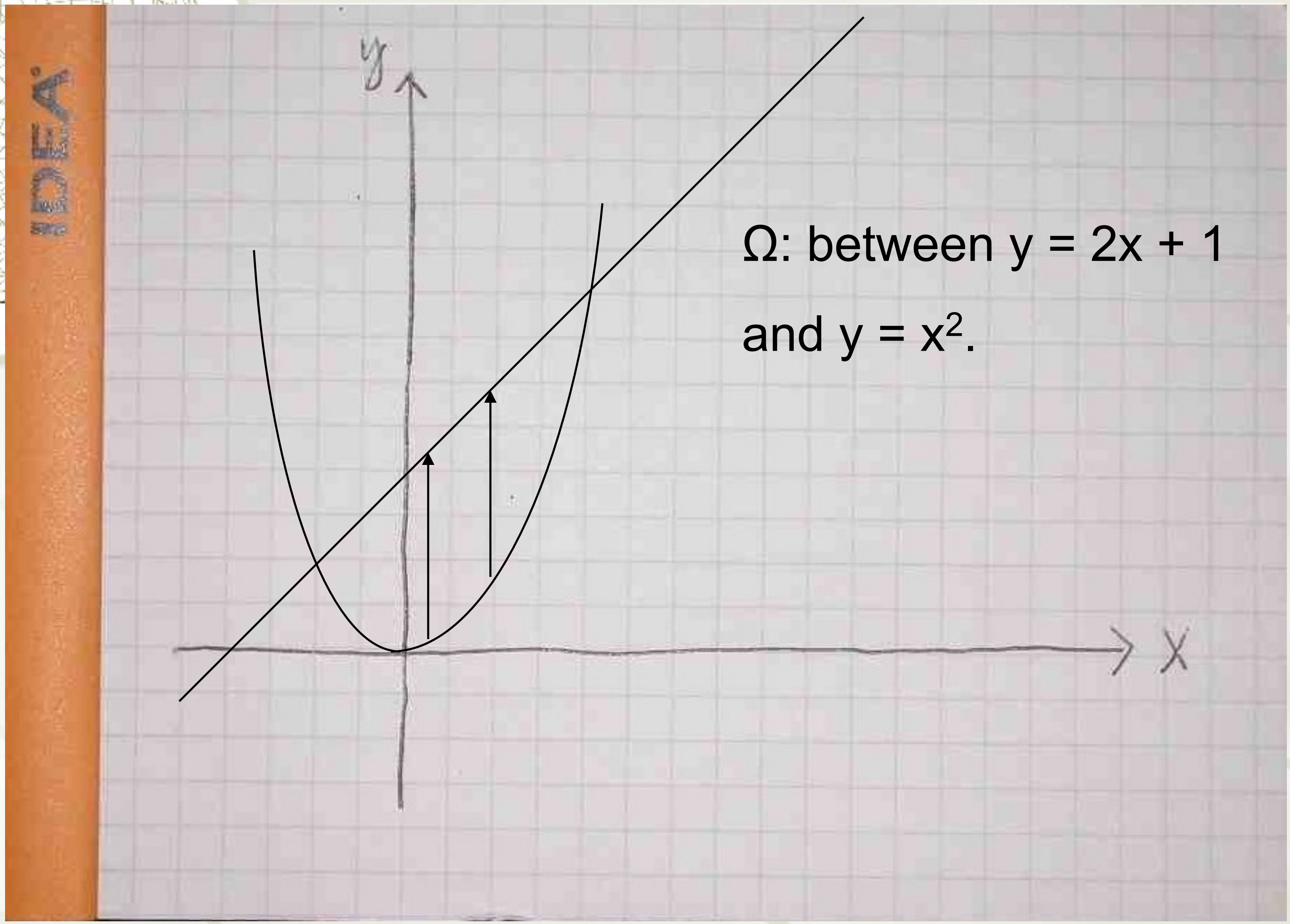

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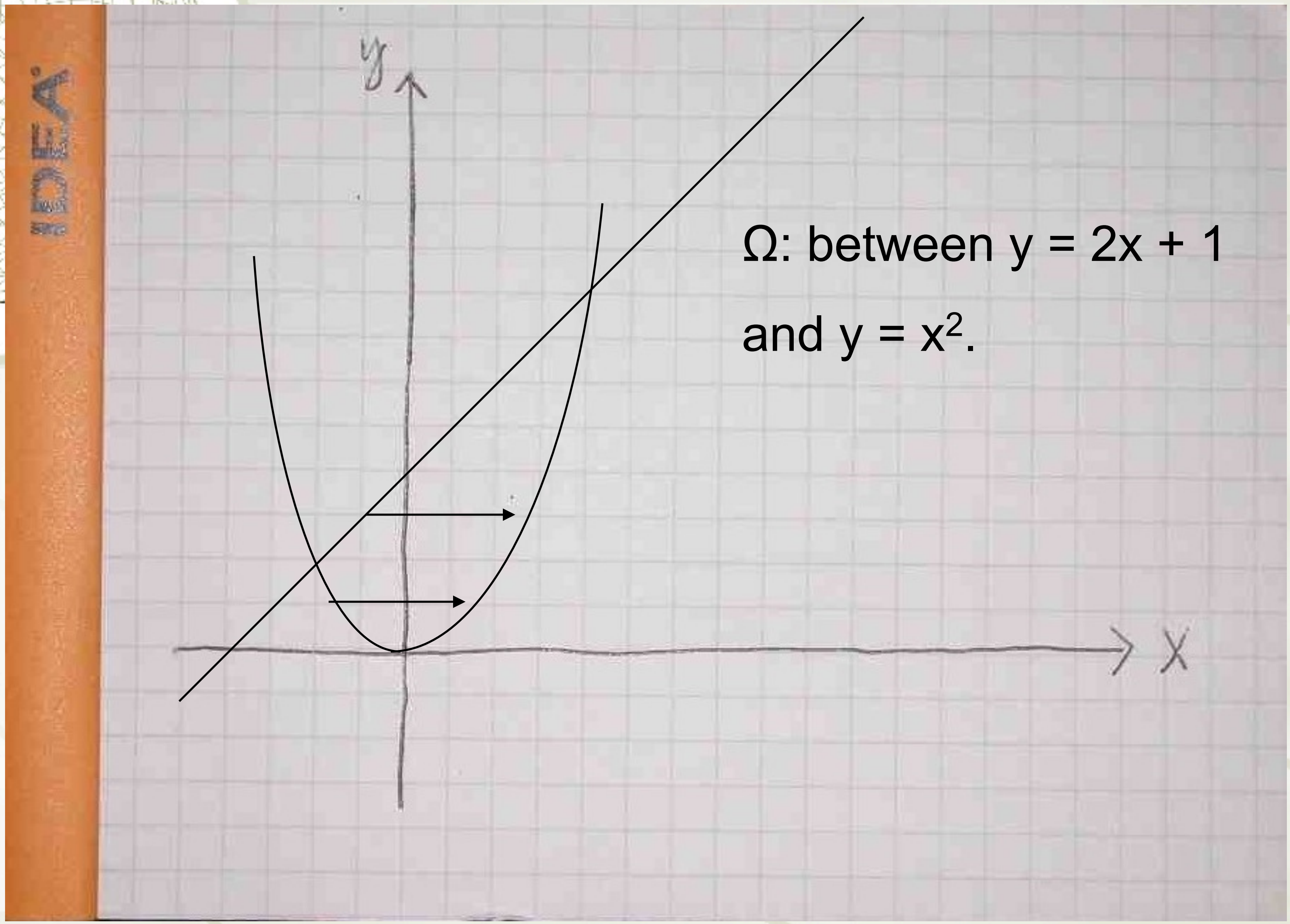

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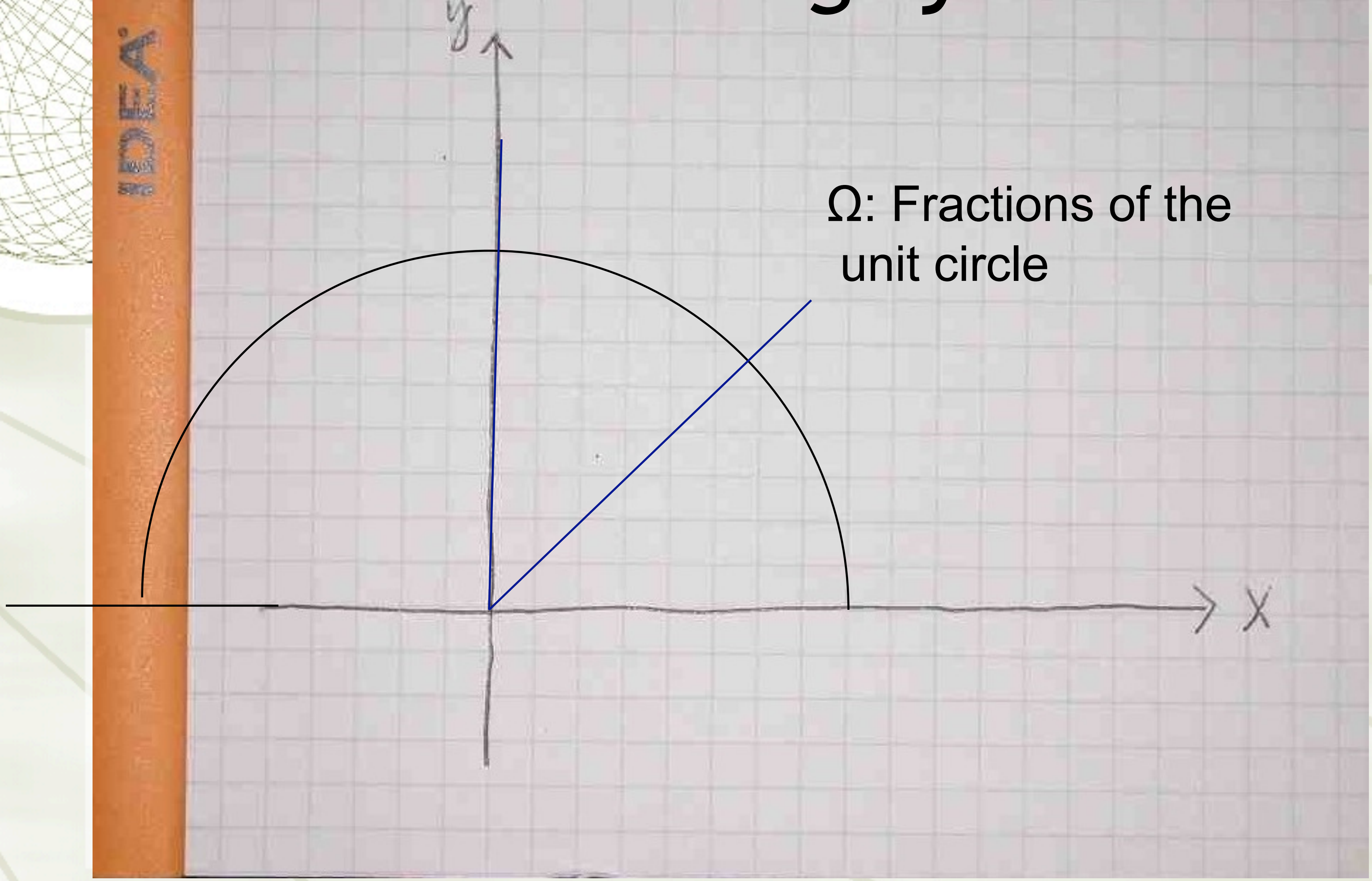
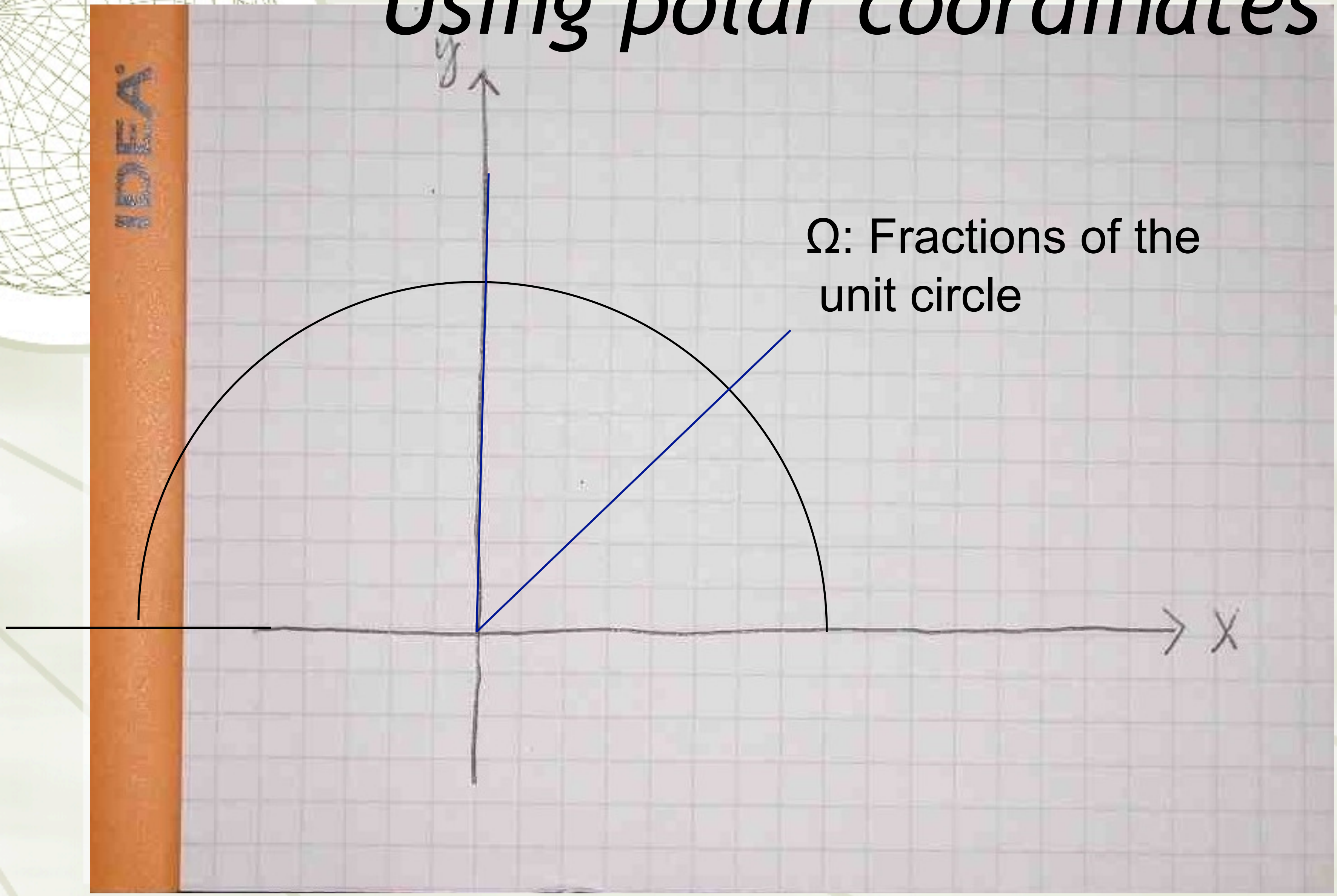
Using symmetries



Ω: Fractions of the unit circle

Using polar coordinates



Differential equations - another reason for double integrals

$$u''(x) = u(x), \quad u(0) = 1, u'(0) = -1,$$

O.D.E.   I.C.

Differential equations - another reason for double integrals

Given

$$u''(x) = u(x), \quad u(0) = 1, u'(0) = -1,$$

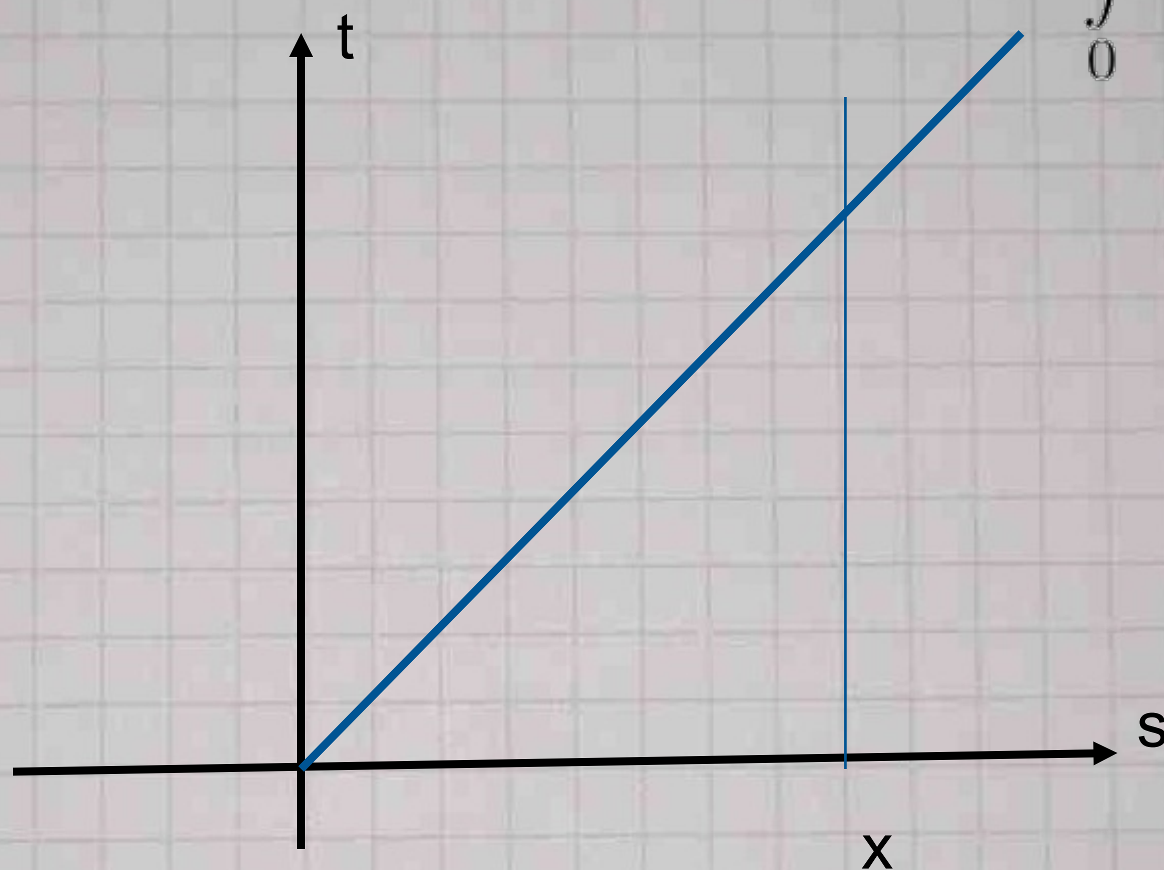
we can attempt a solution by integrating:

$$u'(x) = u'(0) + \int_0^x u''(t)dt = -1 + \int_0^x u(t)dt.$$

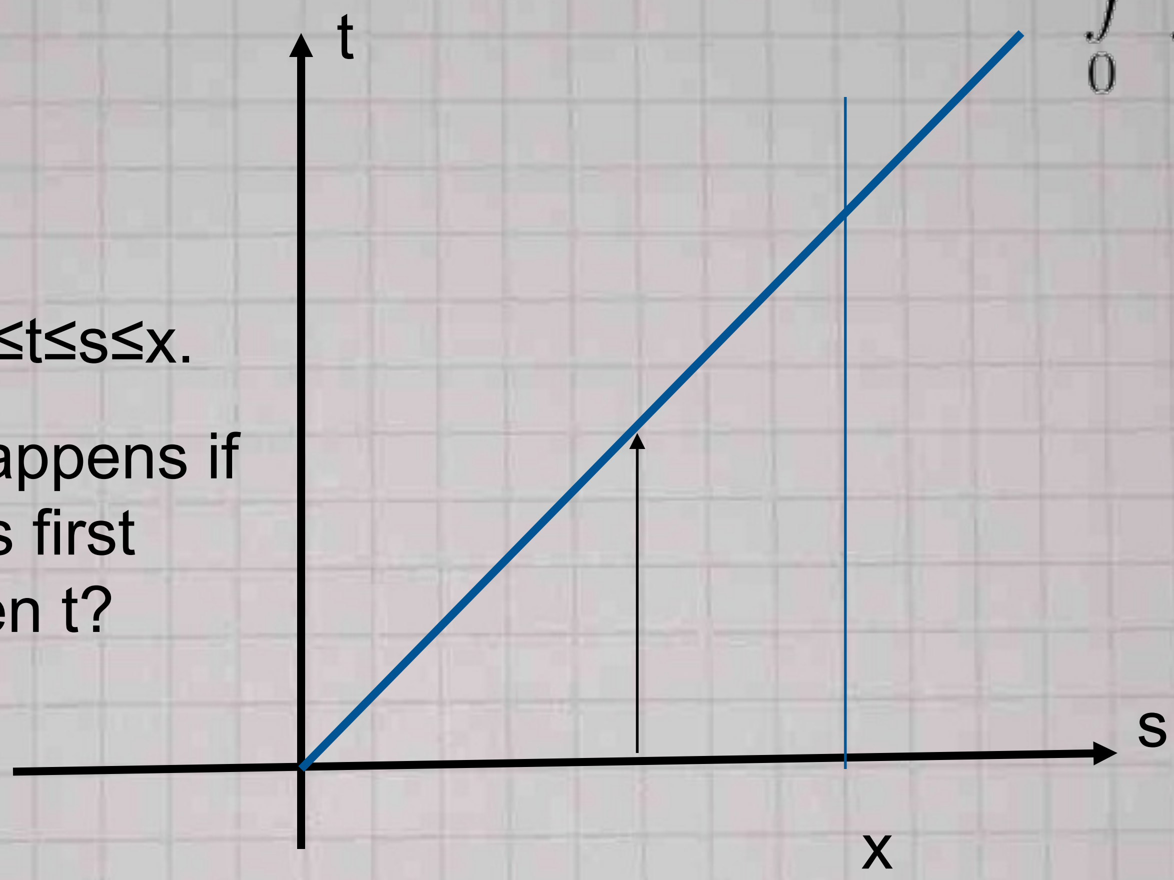
and a second time:

$$u(x) = u(0) + \int_0^x u'(s)ds = 1 - x + \int_0^x \int_0^s u(t)dtds.$$

$$\int_0^x \int_0^s u(t)\,dt\,ds$$



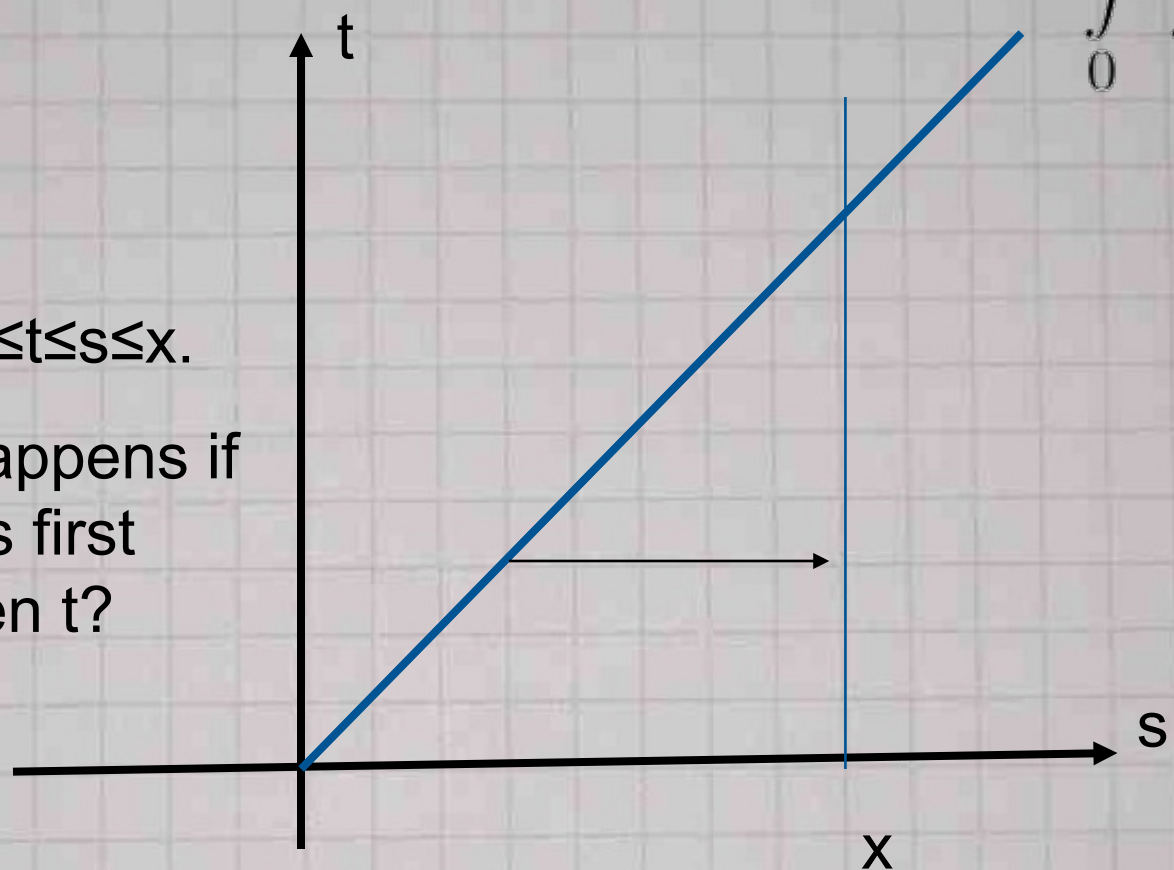
$$\int_0^x \int_0^s u(t)\,dt\,ds$$



So… 0 ≤t≤s≤x.

What happens if we do s first and then t?

$$\int_0^x \int_0^s u(t)\,dt\,ds$$



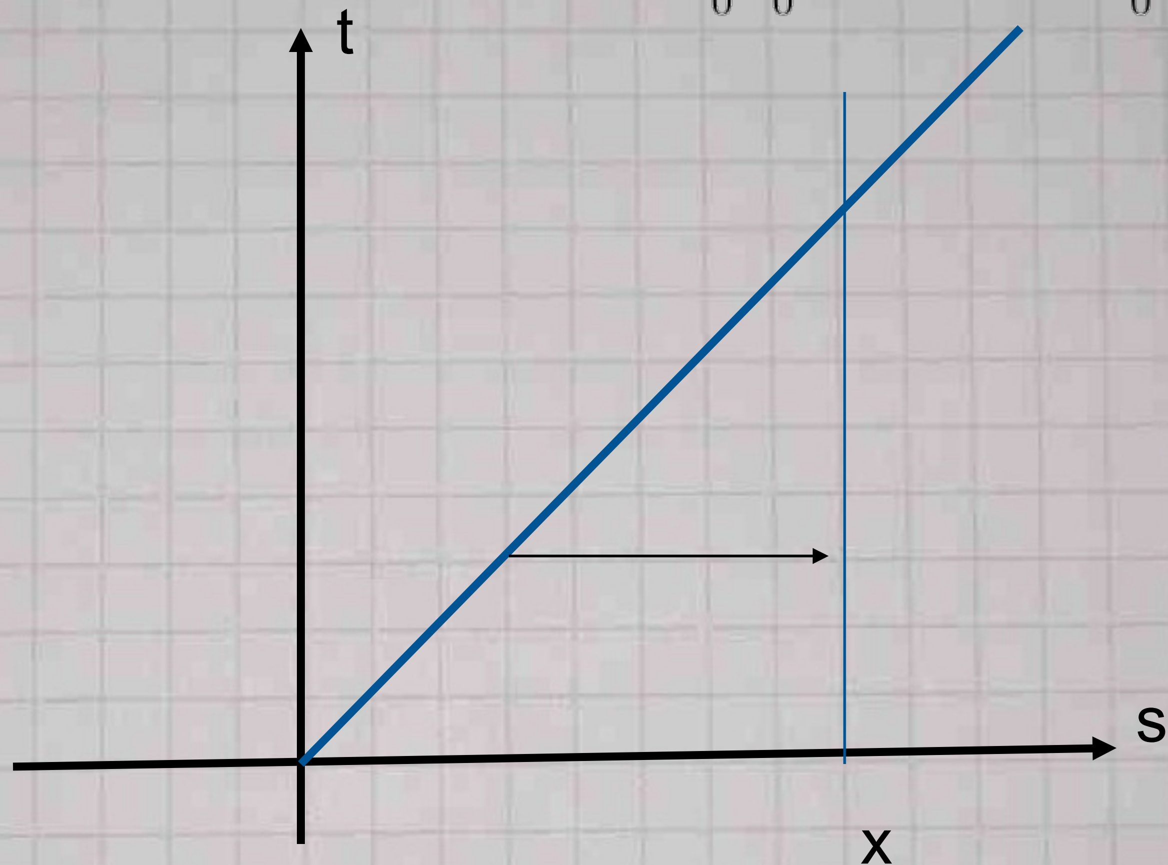
So… 0 ≤t≤s≤x.

What happens if we do s first and then t?

Ans:

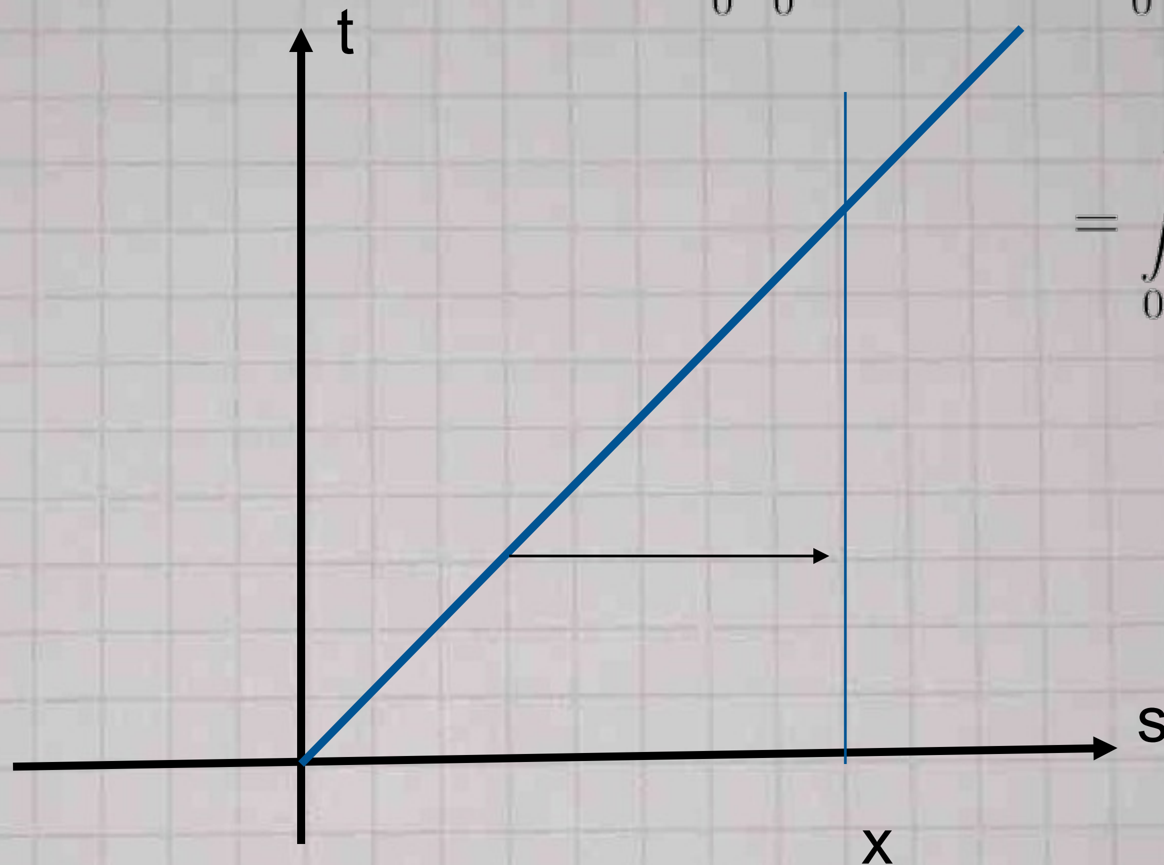
s is between t and x.

$$\int_0^x \int_0^s u(t)\,dt\,ds = \int_0^x \int_t^x u(t)\,ds\,dt.$$



$$\int_0^x \int_0^s u(t)\,dt\,ds = \int_0^x \int_t^x u(t)\,ds\,dt.$$

$$= \int_0^x u(t)(x-t)\,dt.$$



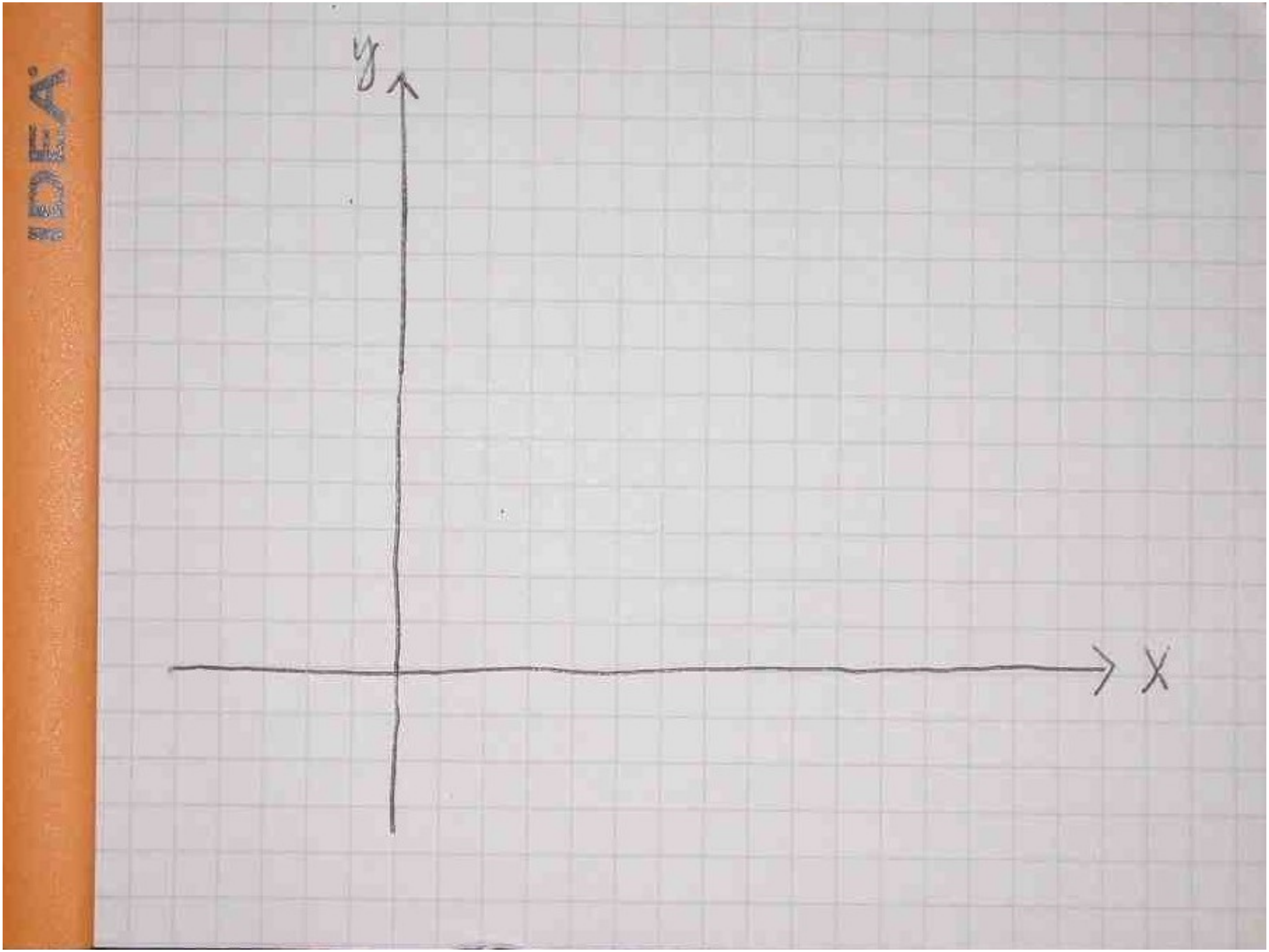
Differential equations - or do you prefer integral equations?

$$u''(x) = u(x), \quad u(0) = 1, u'(0) = -1,$$

If you don't like the differential equation, you can use an integral equation:

$$u(x) = 1 - x + \int_0^x u(t)(x-t)dt$$

- and there's only one integral!


y
x

The End