If you don’t like the limits put on you, change them!

An alternative to try out at homecoming

$e^u\ du/dx$

$e^x\ dx$

cosine secant tangent sine

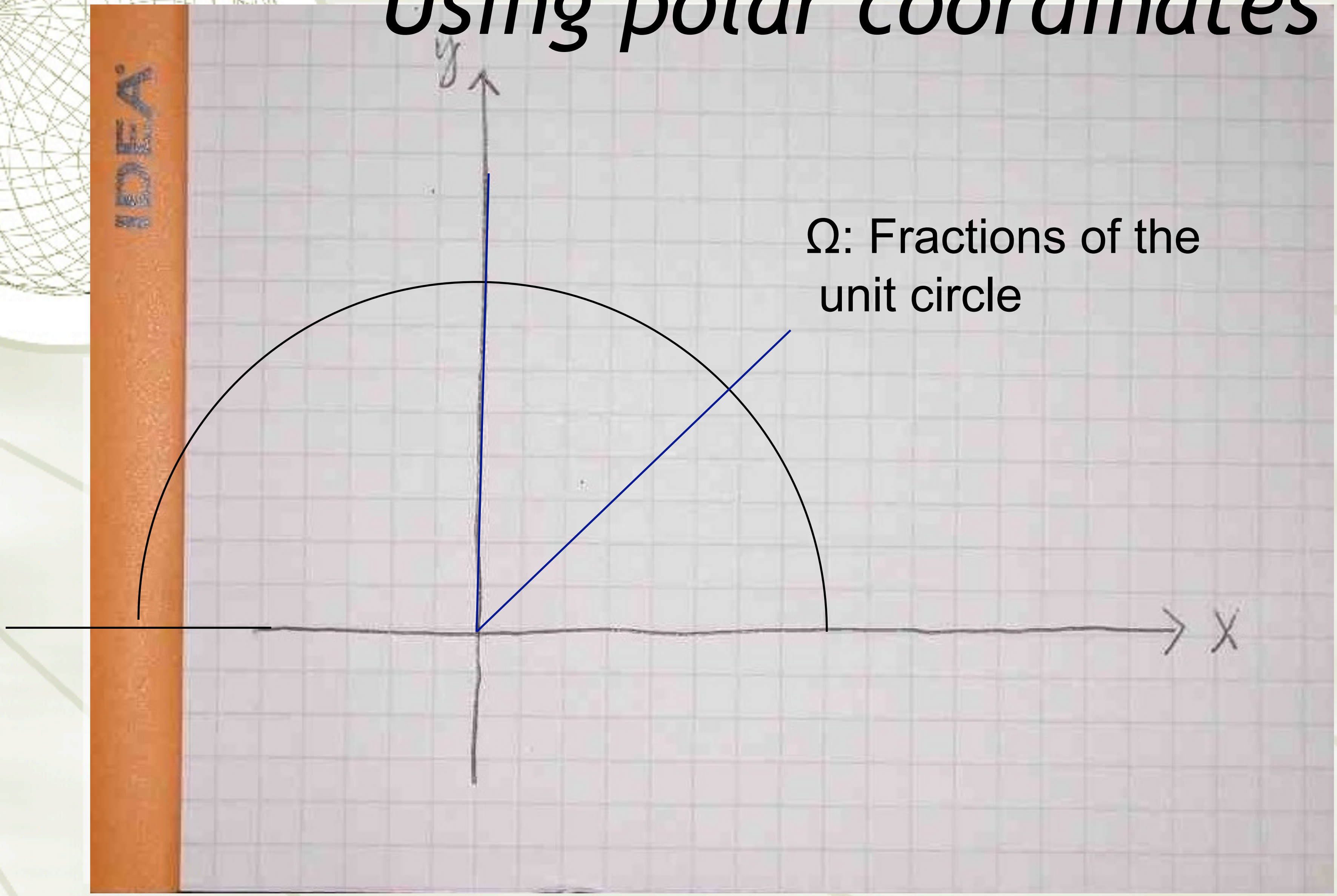
3.14159

integral radical μ dv

slip stick, slide rule, MIT!

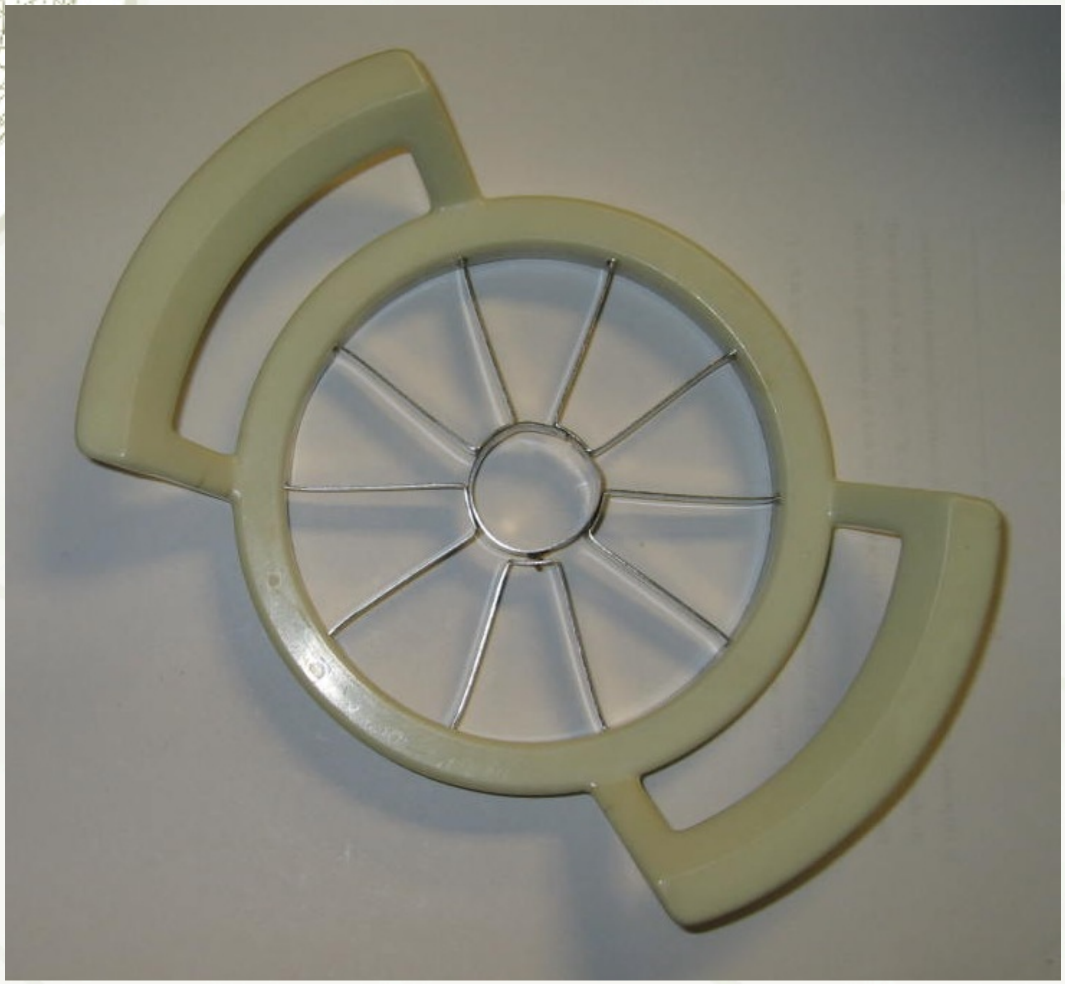
Go-o-o-o-o Tech!

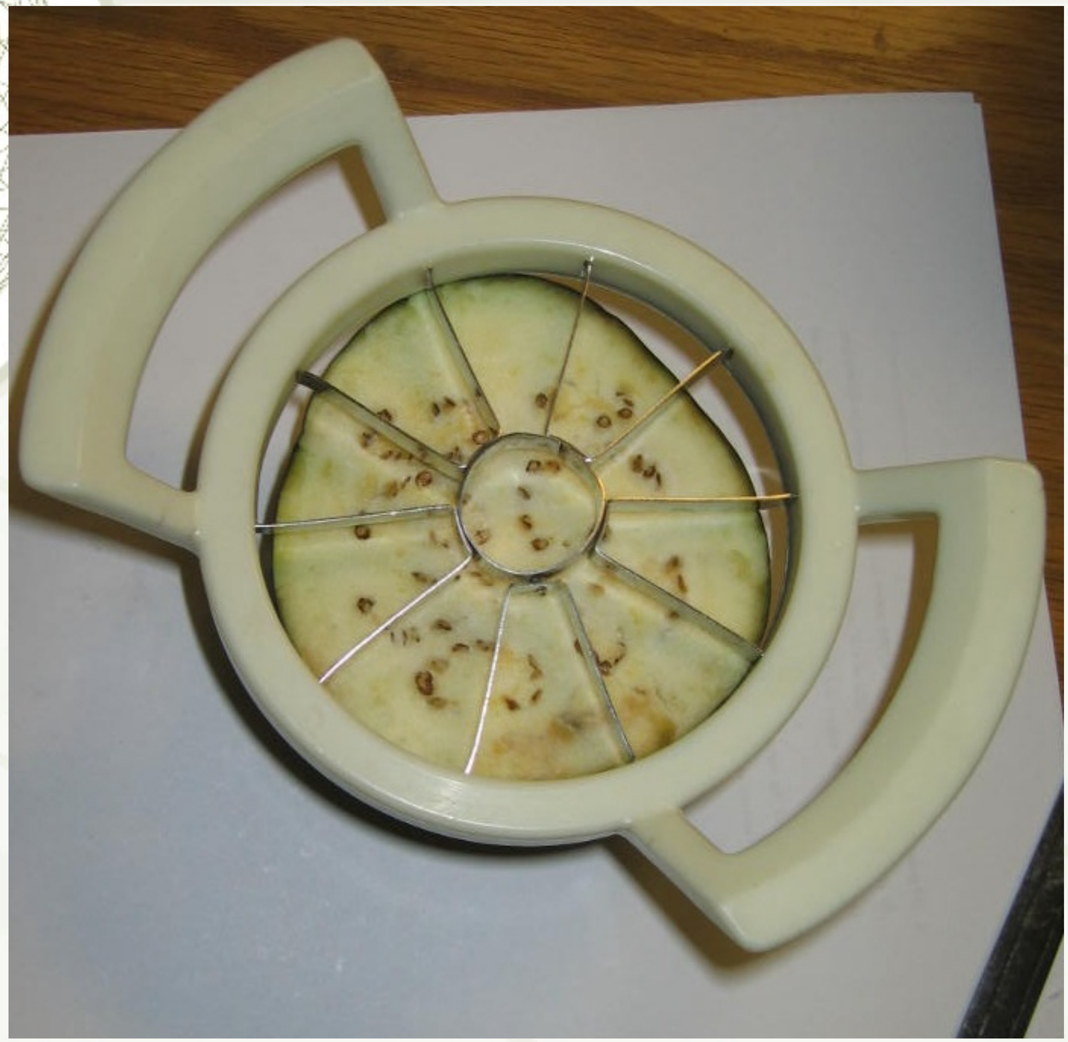
Using polar coordinates

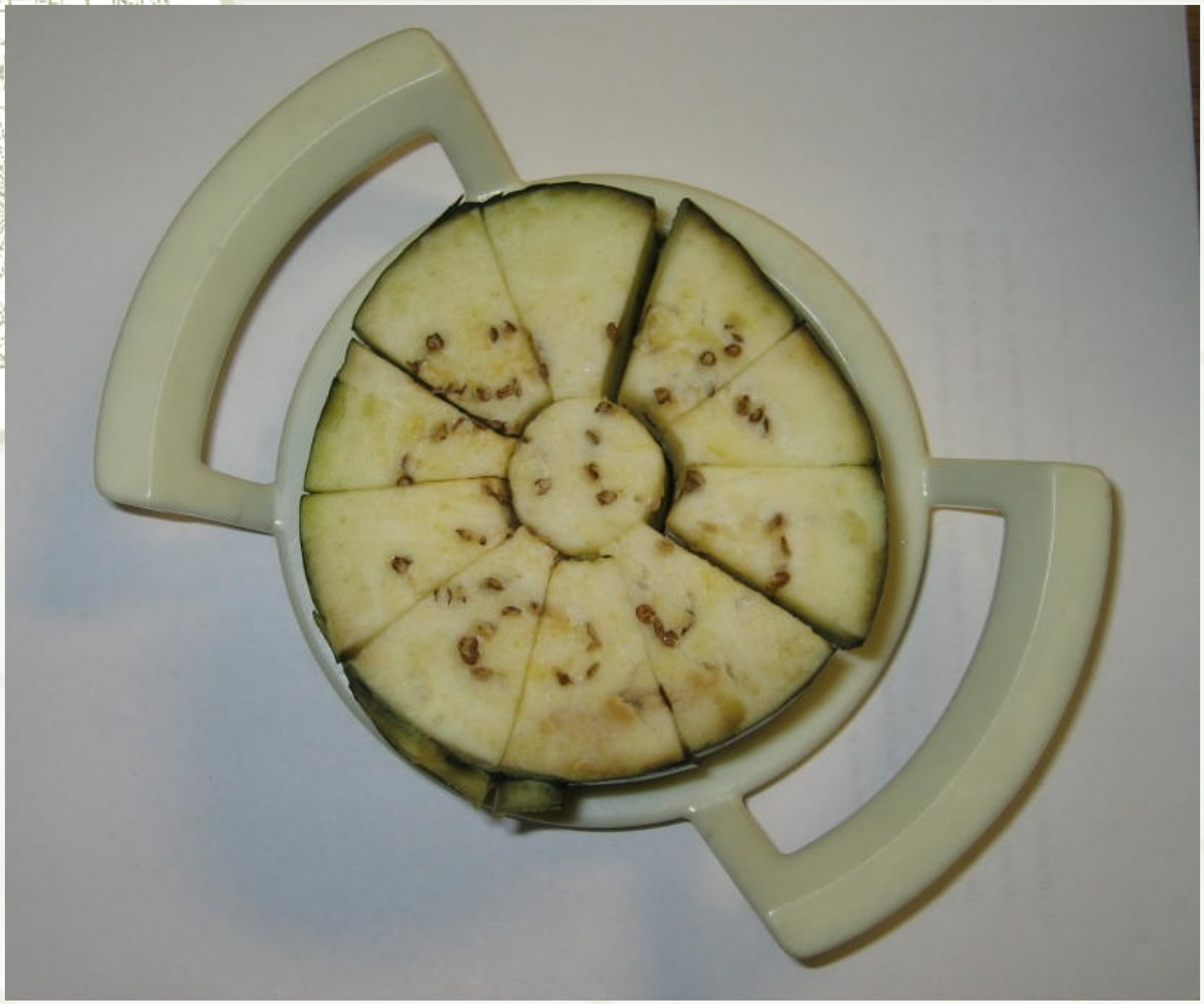


Ω: Fractions of the unit circle

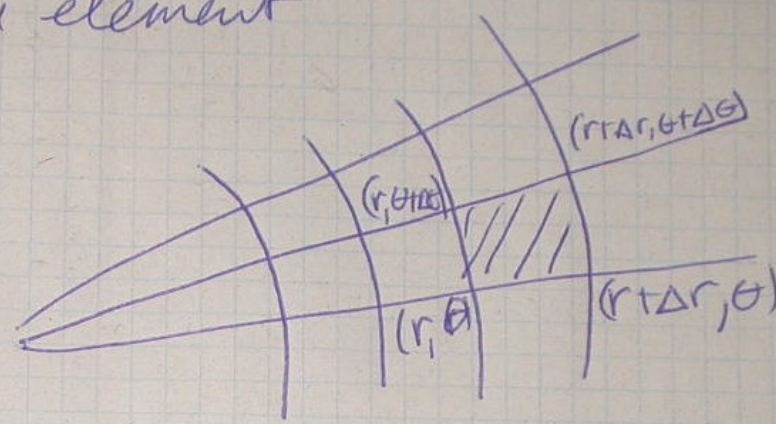
A polar machete







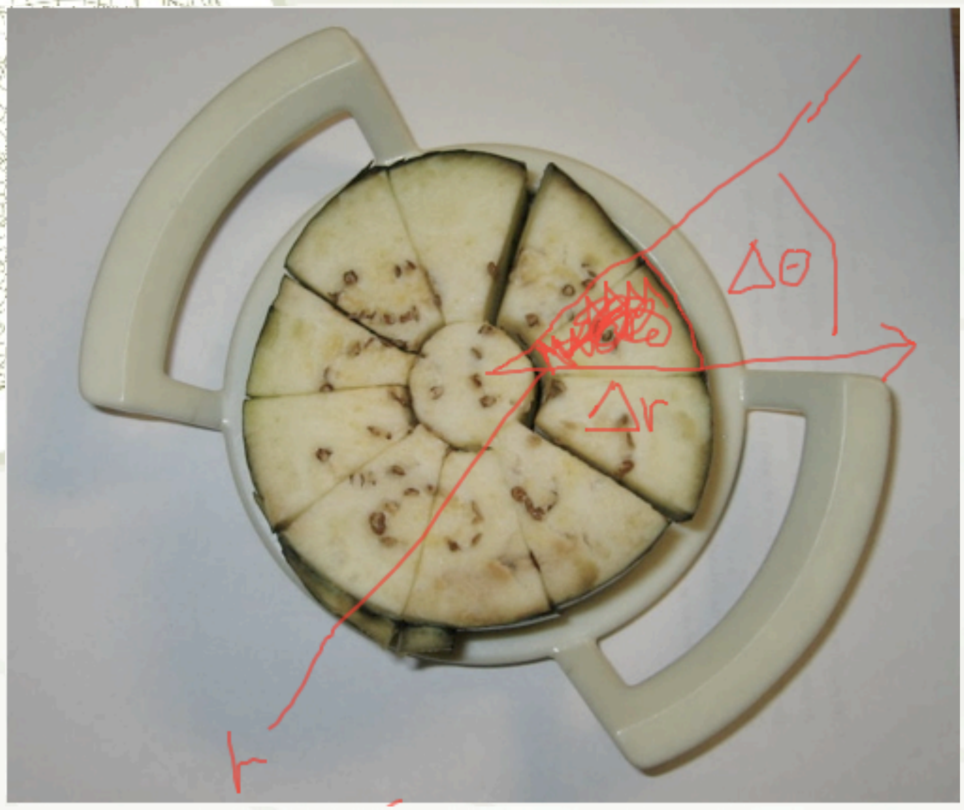
Polar area element



Area from r to $r+\Delta r$ and θ to $\theta+\Delta\theta$ is difference of 2 sectors.

$$A = \pi (r+\Delta r)^2 \frac{\Delta\theta}{2\pi} - \pi r^2 \frac{\Delta\theta}{2\pi}$$

$$= \frac{1}{2}\left(r^2 + 2r\Delta r + (\Delta r)^2 - r^2\right)\Delta\theta = r\Delta r\Delta\theta + \frac{(\Delta r)^2}{2}\Delta\theta$$



There is a cancellation of the r^2's, and when we reduce to the infinitesimal level we neglect $(\Delta r)^2$, leaving:

$$dA = r\,dr\,d\theta.$$

Don't forget the extra factor of r!

Tip: Remember the dimensions. Area is measured in cm^2. Both r and dr have units cm^1.

Example

- Integrate xy over the part of the unit circle in the first quadrant below the line y=x.
- The integrand x y becomes

$$r\cos(\theta)\, r\sin(\theta) = r^2 \sin(2\theta)/2$$

- The limits of integration are

$$0 \le r \le 1,\ 0 \le \theta \le \pi/4.$$

Example - compare Cartesian and polar integrals

- Cartesian calculation:

```
In[5]:= Integrate[x y, {x, y, Sqrt[1 - y^2]}]
```

Out[5]= $\frac{y}{2} - y^3$

```
In[6]:= Integrate[%, {y, 0, 1 / Sqrt[2]}]
```

Out[6]= $\frac{1}{16}$

Don't forget!

- Polar calculation:

```
In[7]:= Integrate[(r^2 Sin[2 theta] / 2) * r, {r, 0, 1}]
```

Out[7]= $\frac{1}{4}$ Cos[theta] Sin[theta]

```
In[8]:= Integrate[%, {theta, 0, Pi / 4}]
```

Out[8]= $\frac{1}{16}$

One of the all-time great integrals

$$\int_0^\infty e^{-x^2} dx = \underline{\hspace{10cm}}$$

One of the all-time great integrals

A really weird approach. First we square the integral, which we'll call I:

$$I^2 = \int_0^{\infty} e^{-x^2} dx \int_0^{\infty} e^{-y^2} dy$$

Right?

One of the all-time great integrals

$$I^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy$$

One of the all-time great integrals

$$I^2 = \int_0^\infty e^{-x^2}dx \int_0^\infty e^{-y^2}dy$$

$$= \int_0^\infty \int_0^\infty e^{-x^2-y^2}dxdy$$

One of the all-time great integrals

$$I^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy$$

$$= \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r dr d\theta$$

$$\int_0^\infty e^{-u}\,du = 1$$

One of the all-time great integrals

$$I = \int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2$$

$$I^2 = \int_0^\infty e^{-x^2}\,dx \int_0^\infty e^{-y^2}\,dy$$

$$= \iint_R e^{-x^2-y^2}\,dx\,dy = \iint_R e^{-r^2}\,r\,dr\,d\theta$$

$$u = r^2, \quad r\,dr = \tfrac{1}{2}du$$

$$= \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,\tfrac{1}{2}dr^2\,d\theta$$

$$= \frac{\pi}{2}\cdot\frac{1}{2}\cdot 1 = \pi/4$$

Averages

Integrate, but divide by the length (1-D), area (2-D), or volume (3-D).

Example: Your pizza has radius 20 cm and at position (x,y) its temperature is

$80 + 2x - (x^2+y^2)/20$. (Celsius) What's the average temperature?

Averages - some tricks

Problem: Find the average of

$$80 + 2x - (x^2+y^2)/20$$

on the pizza.

Trick 1. The integral of f + g is the integral of f + the integral of g. So the ave. of f+g is the ave. of f + the ave. of g. And the ave. of 80 is 80.

Averages - some tricks

Problem: Find the average of

$$80 + 2x - (x^2+y^2)/20$$

on the pizza.

Trick 1. The integral of f + g is the integral of f + the integral of g. So the ave. of f+g is the ave. of f + the ave. of g. And the ave. of 80 is 80.

Trick 2. f(x) = x is an odd function, and this is an even pizza (not changed by x ↔ -x). So…

Averages - some tricks

Problem: Find the average of 80 + 2 x - (x^2+y^2) /20 on the pizza.

Trick 3. Use the right coordinate system. Polar.

$$\frac{1}{\pi 20^2}\int_\Omega \left(-\frac{x^2+y^2}{20}\right) dxdy = -\frac{1}{\pi 20^3}\int_0^{2\pi}\int_0^{20} r^2 \mathbf{r} dr d\theta$$

Averages

Problem: Find the average of 80 + 2 x - (x^2+y^2) /20 on the pizza.

Answer: 80 + 0 + this integral:

$$\frac{1}{\pi 20^2}\int_{\Omega}\left(-\frac{x^2+y^2}{20}\right)dxdy = -\frac{1}{\pi 20^3}\int_0^{2\pi}\int_0^{20} r^2 \mathbf{r} dr d\theta$$

$$= -\frac{1}{\pi 20^3}\int_0^{2\pi}\frac{20^4}{4}d\theta = -10.$$

The average temperature is 70 C.

Differential equations - another reason for double integrals

$$u''(x) = u(x), \quad u(0) = 1, u'(0) = -1,$$

O.D.E.

I.C.

Differential equations - another reason for double integrals

Given

$$u''(x) = u(x), \quad u(0) = 1, u'(0) = -1,$$

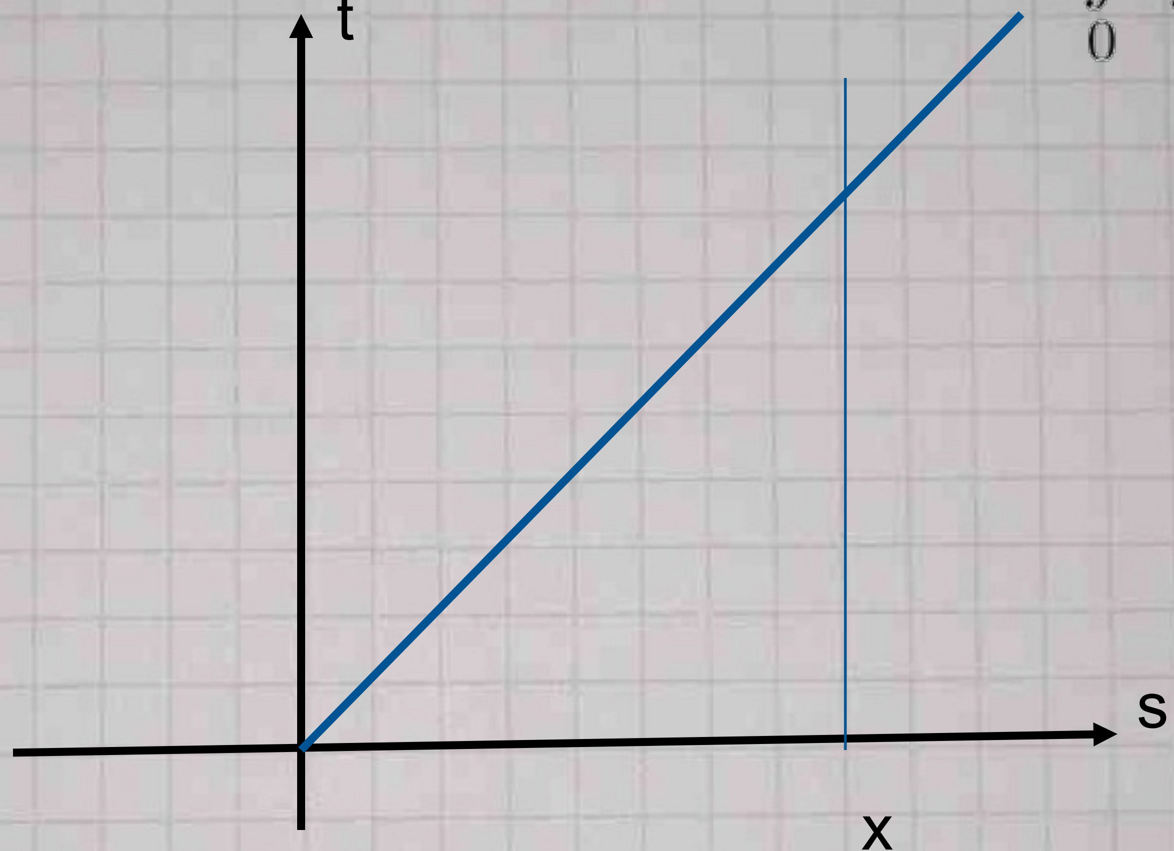
we can attempt a solution by integrating:

$$u'(x) = u'(0) + \int_0^x u''(t)dt = -1 + \int_0^x u(t)dt.$$

and a second time:

$$u(x) = u(0) + \int_0^x u'(s)ds = 1 - x + \int_0^x \int_0^s u(t)dtds.$$

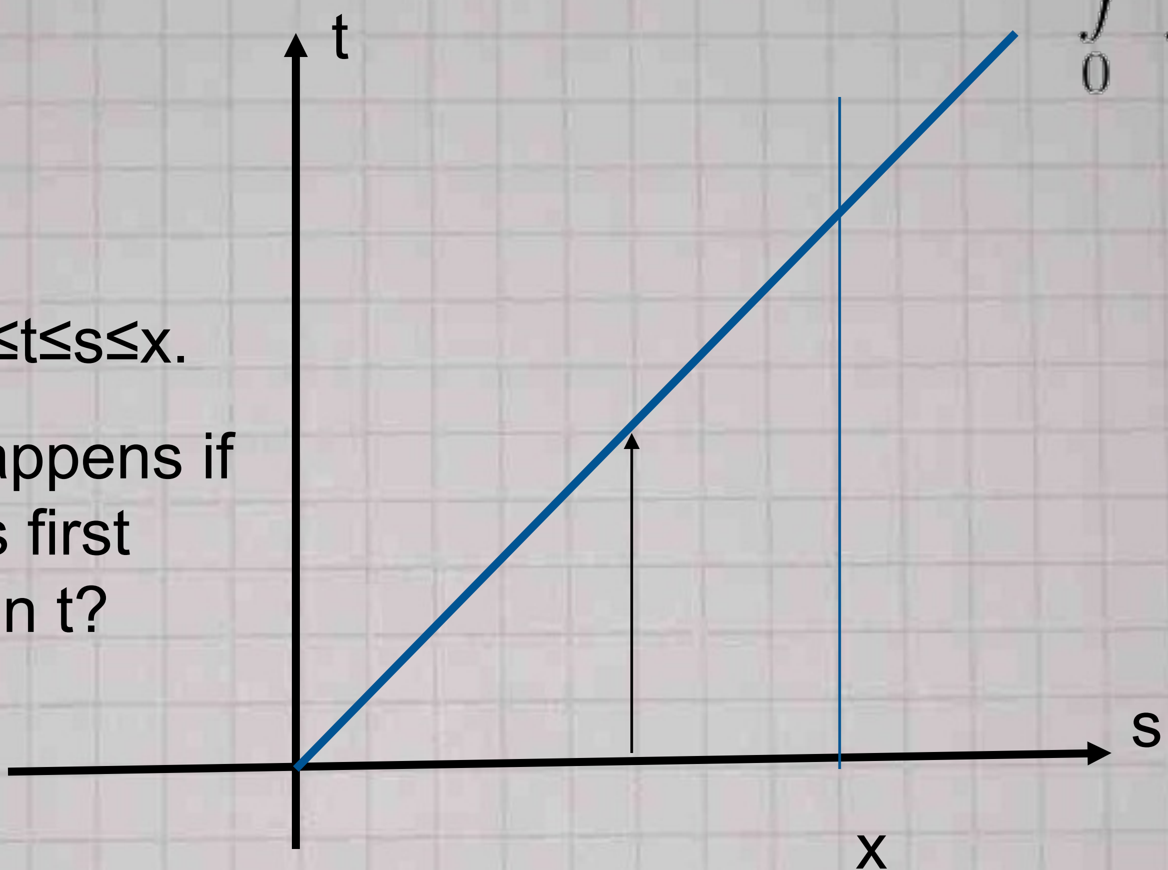
$$\int_0^x \int_0^s u(t)\,dt\,ds$$



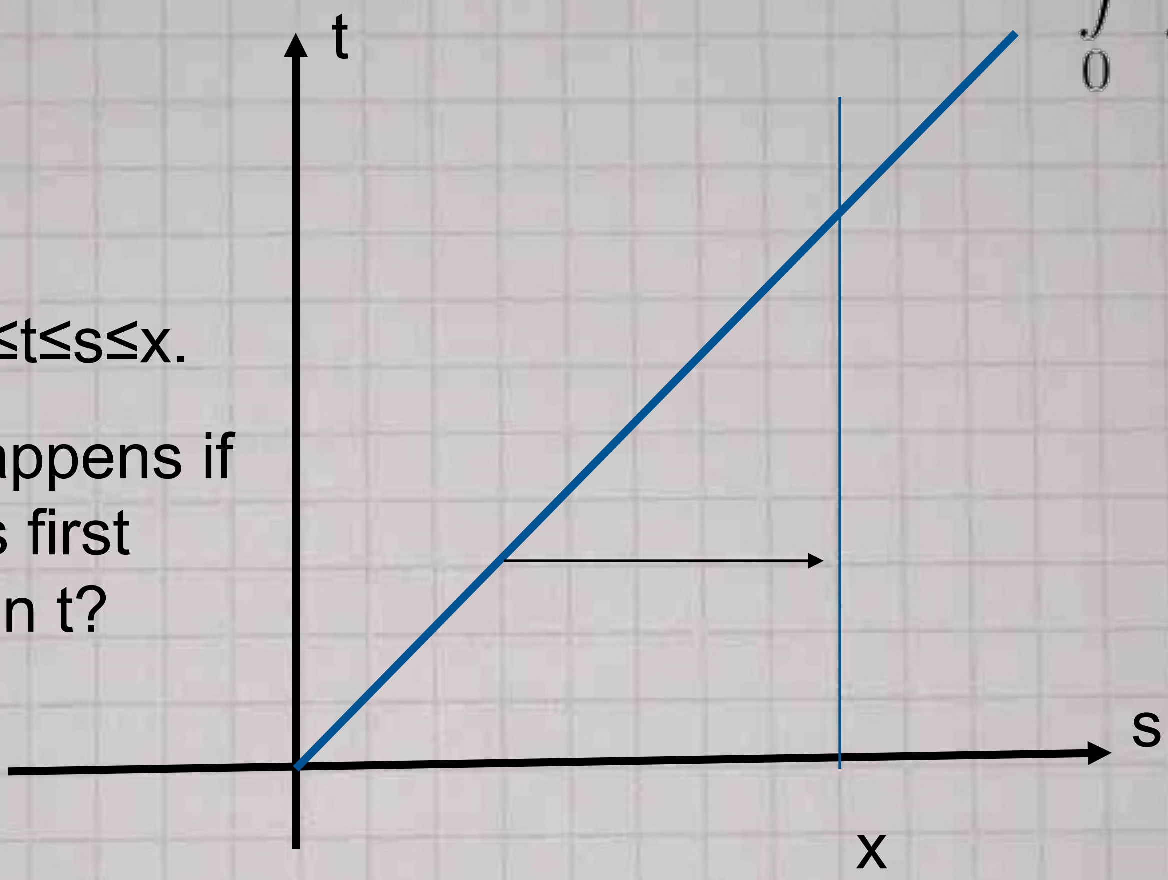
$$\int_0^x \int_0^s u(t)\,dt\,ds$$

So… 0 ≤t≤s≤x.

What happens if we do s first and then t?



$$\int_0^x \int_0^s u(t)\,dt\,ds$$



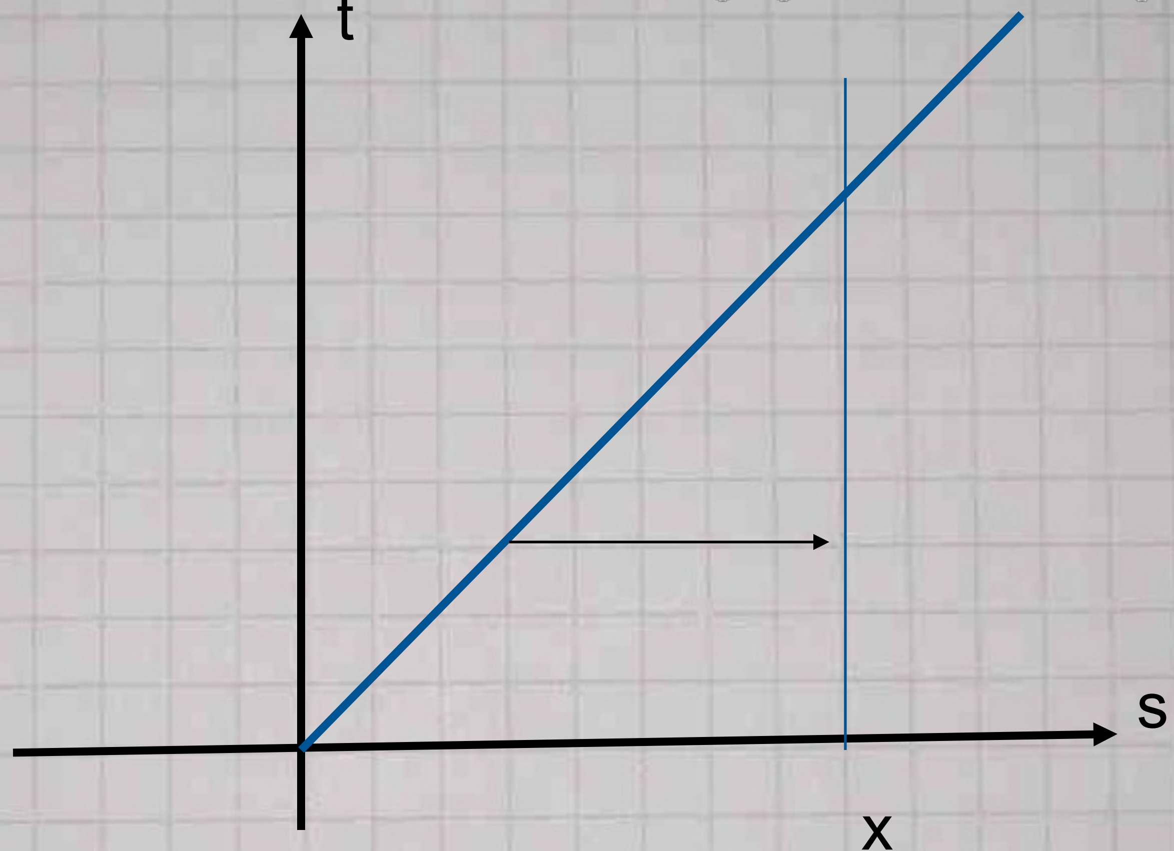
So… $0 \le t \le s \le x$.

What happens if we do s first and then t?

Ans:

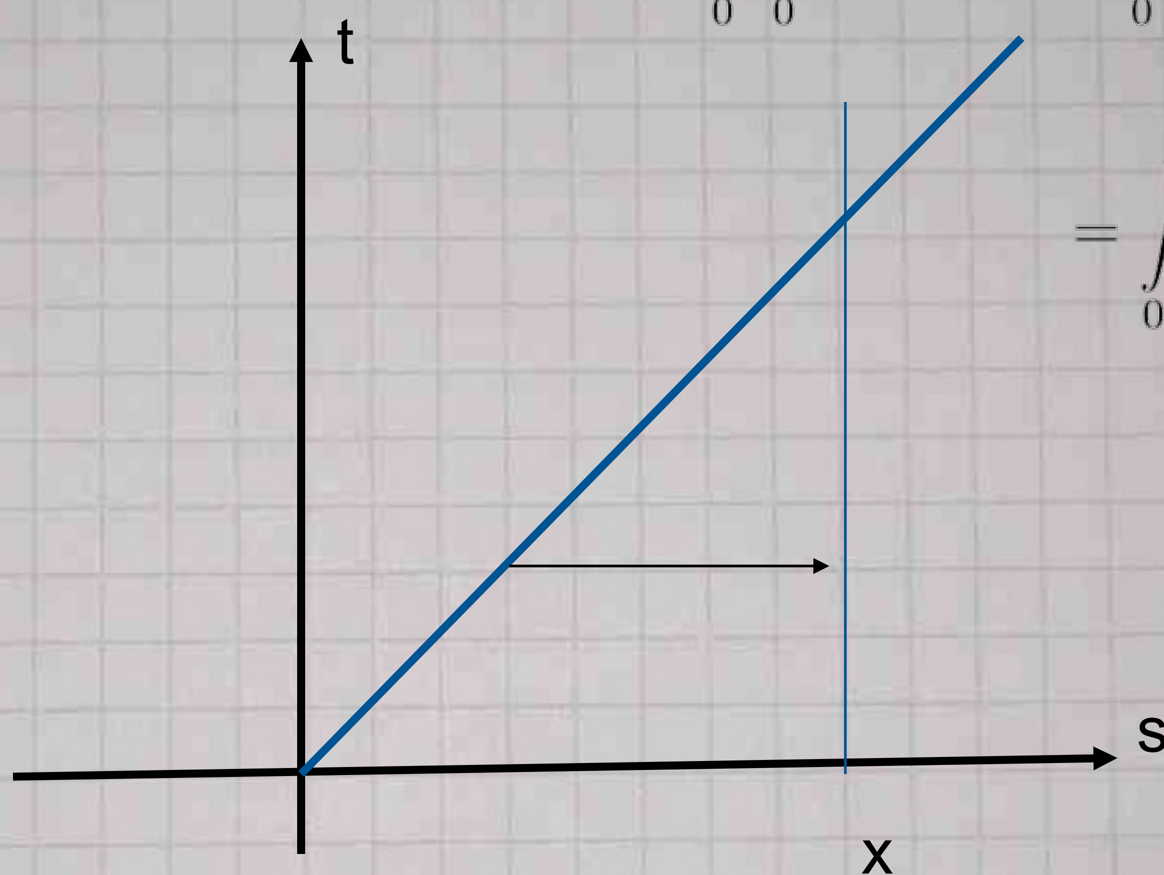
s is between t and x.

$$\int_0^x \int_0^s u(t)\,dt\,ds = \int_0^x \int_t^x u(t)\,ds\,dt.$$



$$\int_0^x \int_0^s u(t)\,dt\,ds = \int_0^x \int_t^x u(t)\,ds\,dt.$$

$$= \int_0^x u(t)(x-t)\,dt.$$



Differential equations - or do you prefer integral equations?

$$u''(x) = u(x), \quad u(0) = 1, u'(0) = -1,$$

If you don't like the differential equation, you can use an integral equation:

$$u(x) = 1 - x + \int_0^x u(t)(x-t)dt$$

- and there's only one integral!

The End