A fine balance

About the tests….

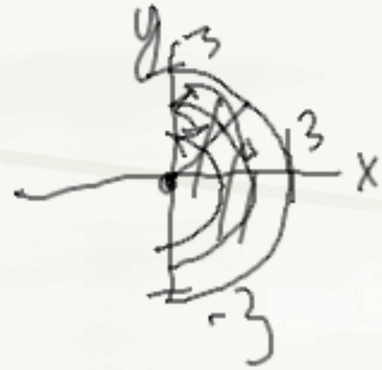
- Has been graded and reviewed. Median 73.
- More than usual typos. Sorry!
  1. We adjusted points on one problem and checked for systematic differences between the versions before averaging.
  2. Next time the teaching team will prepare solutions farther in advance.

Lowest points on polar integral

5. (A clone of 17.4 #9) Calculate

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}} \sqrt{4x^2+4y^2}\,dx\,dy = \underline{\hspace{6cm}}$$

Lowest points on polar integral



5. (A clone of 17.4 #9) Calculate

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}} \sqrt{4x^2+4y^2}\,dx\,dy = \underline{\hspace{6cm}}$$

$$= \int_0^3 \int_{-\pi/2}^{\pi/2} 2r \boxed{r\,d\theta\,dr}$$

$$= 2\int_0^3 r^2\,dr \int_{-\pi/2}^{\pi/2} d\theta = 2\,\frac{1}{3}\,3^3\,\pi = \boxed{18\pi}$$

Low points on limit switcheroo

6. Let $I := \int_\Omega \int (x + 2y - 3)dxdy$, where Ω is the region bounded by $y = 2x^2$ and $y = 2$.

a) Write I as an iterated integral where x is integrated first:

$$\int_{\_\_\_\_\_}^{\_\_\_\_\_} \int_{\_\_\_\_\_}^{\_\_\_\_\_} \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ d\_\_\_ d\_\_\_$$

b) Write I as an iterated integral where y is integrated first:

$$\int_{\_\_\_\_\_}^{\_\_\_\_\_} \int_{\_\_\_\_\_}^{\_\_\_\_\_} \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ d\_\_\_ d\_\_\_$$

c) Evaluate I = ____________________.

Low points on limit switcheroo

$-\sqrt{\frac{y}{2}} \le x \le \sqrt{y/2}$

6. Let $I := \int_\Omega \int (x+2y-3)\,dx\,dy$, where Ω is the region bounded by $y = 2x^2$ and $y = 2$.

a) Write I as an iterated integral where x is integrated first:

$$\int_{y=0}^{y=2} \int_{-\sqrt{y/2}}^{x=\sqrt{y/2}} (x+2y-3)\; dx\; dy$$

b) Write I as an iterated integral where y is integrated first:

$$\int_{-1}^{+1} \int_{2x^2}^{2} (x+2y-3)\; dy\; dx$$

c) Evaluate $I =$ ______________________.

Speaking of tests….

- The final exam is scheduled for Period 5, Tuesday, 9 December, 11:30-2:20.

  - Potential conflict with Physics 2212
  - Potential conflict with CS 1371 (Section E)

  I have been in contact with Prof. Murray about the conflict with Physics 2212. Write him by 10 Nov.
  Write me about other conflicts ASAP!

Center of mass and centroid

What is the point at which a “plate” - a 2-D region - will balance?

Ans: The average position.

Case 1. The density is constant. Then the balancing point is also called the *centroid*.

Center of mass and centroid

Case 1. The density is constant. Then the balancing point is also called the *centroid*.

$$\mathbf{r}_M = \frac{\int\int_\Omega \mathbf{r} dxdy}{Area}$$

That is,

$$x_M = \frac{\int\int_\Omega x dxdy}{\int\int_\Omega dxdy},$$

$$y_M = \frac{\int\int_\Omega y dxdy}{\int\int_\Omega dxdy}.$$

Center of mass and centroid

Case 2. Variable density.

$$\mathbf{r}_M = \frac{\int\int_\Omega \mathbf{r}\lambda(x,y)dxdy}{Mass}$$

Or,

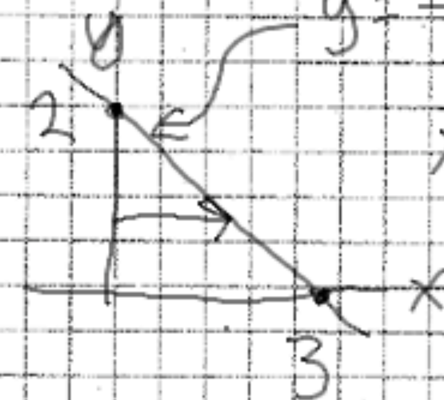
$$x_M = \frac{\int\int_\Omega x\lambda(x,y)dxdy}{\int\int_\Omega \lambda(x,y)dxdy},$$

$$y_M = \frac{\int\int_\Omega y\lambda(x,y)dxdy}{\int\int_\Omega \lambda(x,y)dxdy}.$$

Center of mass and centroid

Examples.

1. Centroid of a triangle.



$y = -\frac{2}{3}x + 2$

$x = -\frac{3}{2}y + 3$

$$X_M = \frac{\int_0^2 \int_0^{3-3y/2} x \, dx \, dy}{A(\triangle)} \qquad \frac{1}{2} 3 \cdot 2 = 3$$

$$\frac{1}{2}x^2 \Big|_0^{3-3y/2} = \frac{1}{2}\left(9 + \frac{9}{4}y^2 - 9y\right)$$

$$\frac{9}{2}\int_0^2 \left(1 + \frac{y^2}{4} - y\right) dy = 9 + \frac{9}{2 \cdot 3 \cdot 4} 2^3 - \frac{9}{2} \frac{1}{2} 2^2 = 3$$

Center of mass and centroid

Examples.

u-

2. Square with density y^2.

$$\frac{\int_0^1\int_0^1 y^2 x \, dx\,dy}{\boxed{\int_0^1\int_0^1 y^2 \, dx\,dy}} = X_m$$

$1/6$

$\frac{1}{3} =$

$$X_m = \frac{1/6}{1/3} = \frac{1}{2}$$

$$Y_m = \frac{1/4}{1/3} = 3/4$$

$$\frac{\int_0^1\int_0^1 y^3 \, dx\,dy}{\int_0^1\int_0^1 y^2 \, dx\,dy} = Y_m$$

Center of mass and centroid

Examples.

3. Centroid of a hemisphere
   (3D example).

Center of mass and centroid

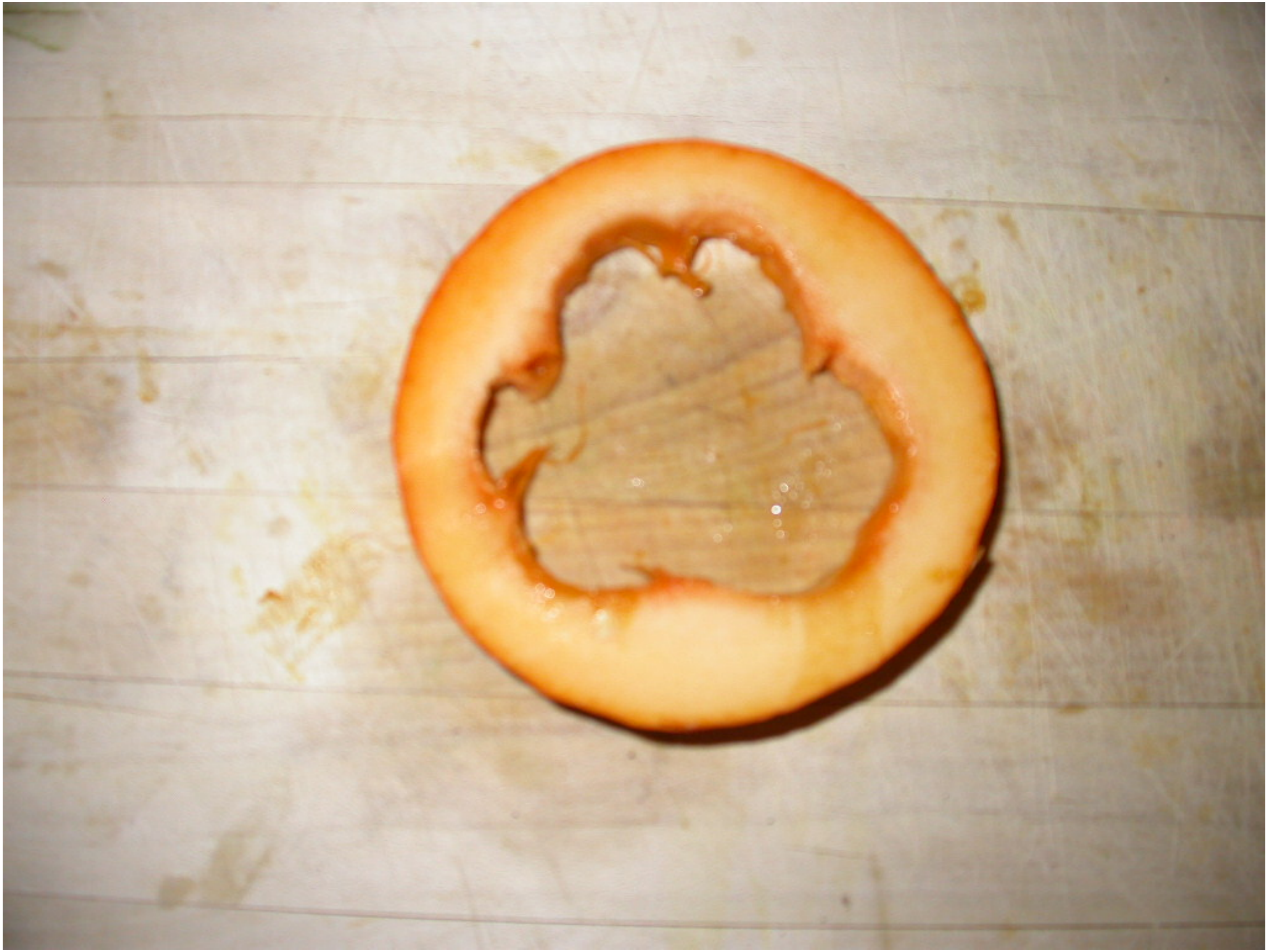
Examples.

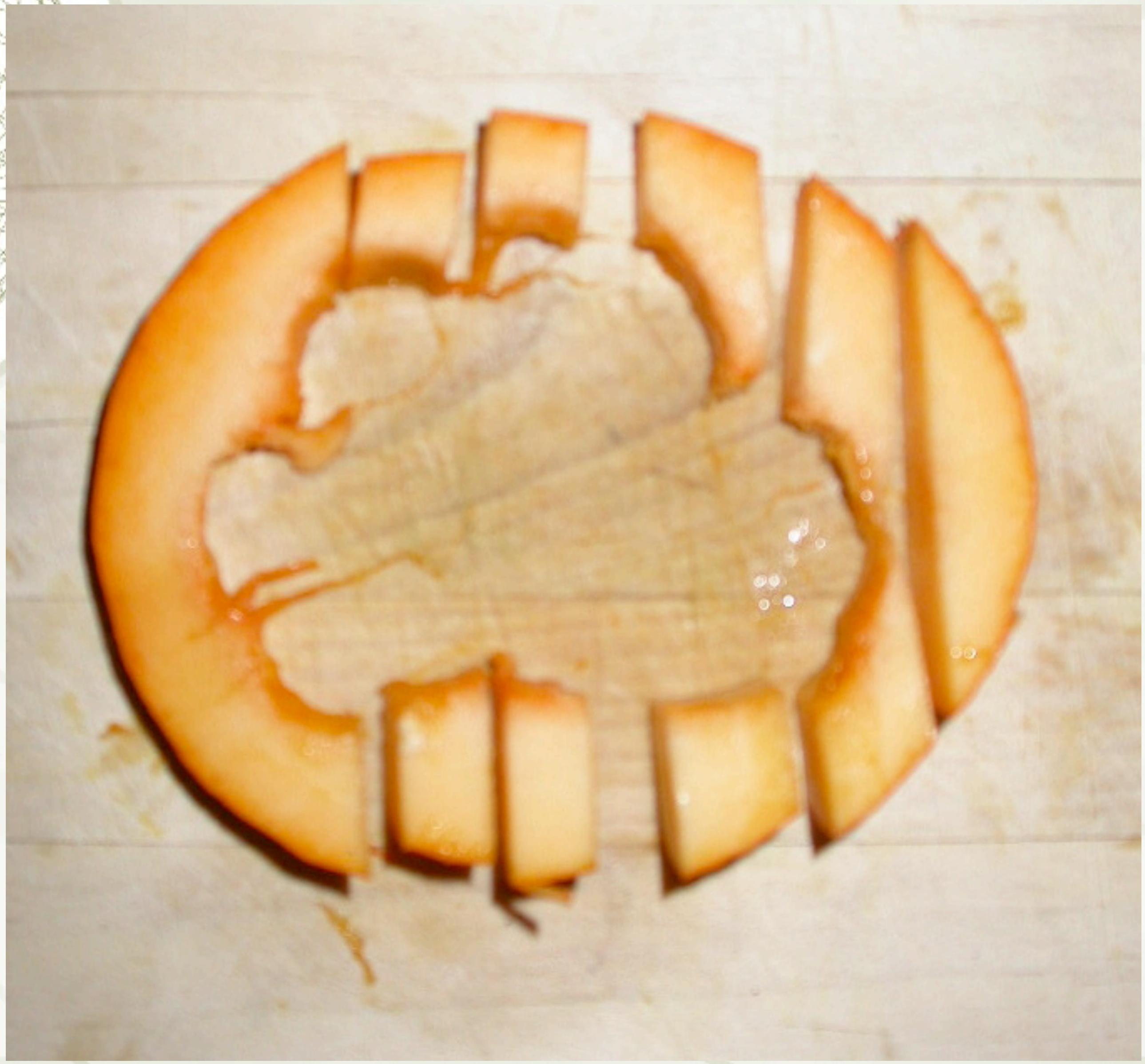
3. Centroid of a hemisphere (3D example).

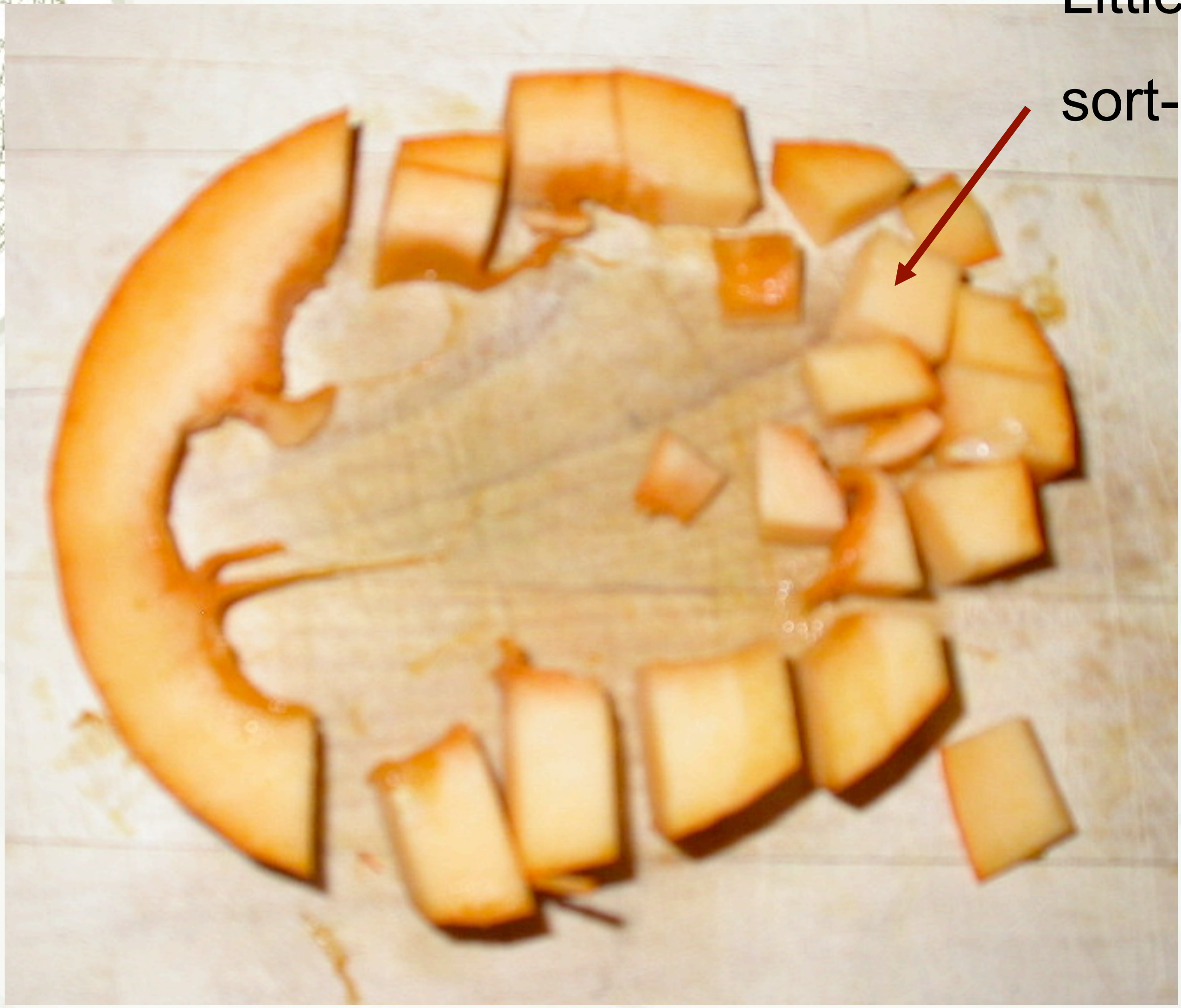
OOPS! Pictures of this important example done in class are “404 not found.”

Triple integrals

- Volume and mass if the height function f(x,y) is not so simple.
- Even if the French fries are not so convenient, you can still *dice* the veggie.







Little cubes,

sort-of

Triple integrals

- Volume: $\int dx\, dy\, dz$

- Mass or other integrals:
  $\int \lambda(x,y,z) dx\, dy\, dz$

- Average:
  $(1/\mathrm{Vol}(\Omega)) \bullet \int \lambda\, (x,y,z) dx\, dy\, dz$

Triple integrals

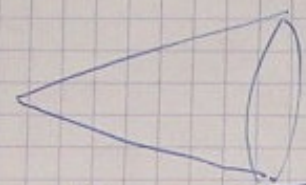
- Examples:
  - Cone $y^2 = x^2 + z^2$ Volume? Centroid?
  - Wedge $0 \le x,y,z$, $x + 2y + 3z \le 3$
  - Integral of $\sin x \sin y \sin z$ with Ω bounded by $z = y$, $z = 0$, $x = 0$, $x = \pi/2$, $y = \pi$.
  - Integral of $y^2 x^2 z$ with $0 \le z \le x^2 - y^2$, $0 \le x \le 1$.

Triple integrals

- Examples:
  - Cone $y^2 = x^2 + z^2$ Volume? Centroid?

$$\int_{\rule{1.5cm}{0.4pt}}^{\rule{1.5cm}{0.4pt}} \int_{\rule{1.5cm}{0.4pt}}^{\rule{1.5cm}{0.4pt}} \int_{\rule{1.5cm}{0.4pt}}^{\rule{1.5cm}{0.4pt}} \rule{3cm}{0.4pt}\, d\rule{1.5cm}{0.4pt}\, d\rule{1.5cm}{0.4pt}\, d\rule{1.5cm}{0.4pt}.$$

Volume of the cone $0 \le y \le 2$, $x^2+z^2 \le y^2$



$$\text{Vol} = \int_0^2 \int_{-y}^{y} \int_{-\sqrt{y^2-x^2}}^{+\sqrt{y^2-x^2}} 1 \; dz\,dx\,dy = \int_0^2 \int_{-y}^{y} 2\sqrt{y^2-x^2}\,dx\;dy$$

$$= \int_0^2 \pi y^2\,dy = \frac{8\pi}{3}$$

Centroid:

$$y_m = \frac{3}{8\pi} \int_0^2 \int_{-y}^{y} \int_{-\sqrt{y^2-x^2}}^{\sqrt{y^2-x^2}} y \; dz\,dx\,dy = \frac{3}{8\pi} \int_0^2 y \cdot \pi y^2\,dy$$

$$= \frac{3}{32}\, 2^4 = \frac{3}{2}$$

$\frac{1}{4}$ of the way from the fat end to the tip.