MATH 2401 - Harrell

Vectors in motion

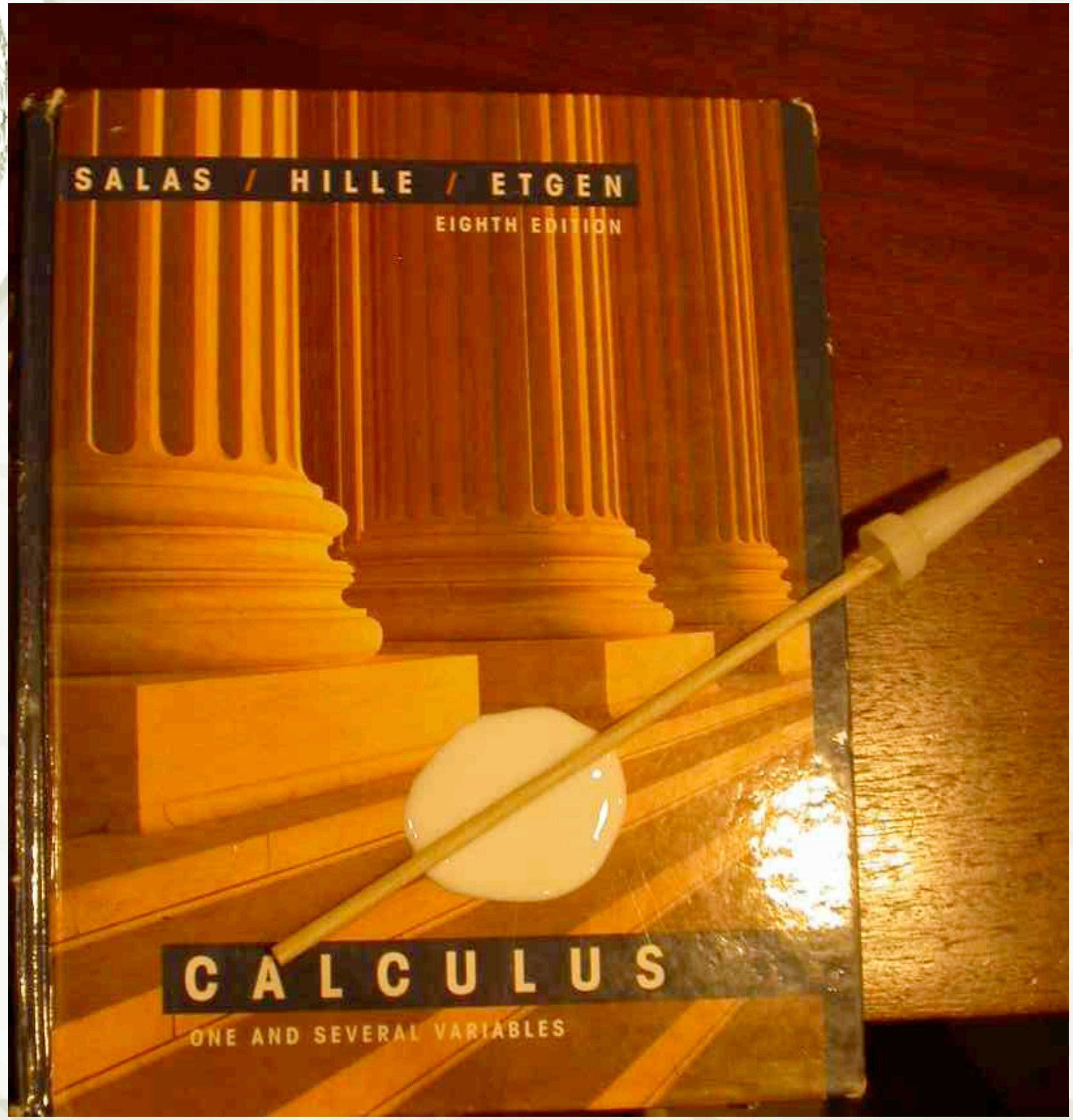
- Uncle Si -
John Saylor Coon, 1854-1938,
Founder of GT School of
Mechanical Engineering,

"Engineering is common sense first, and mathematics next."

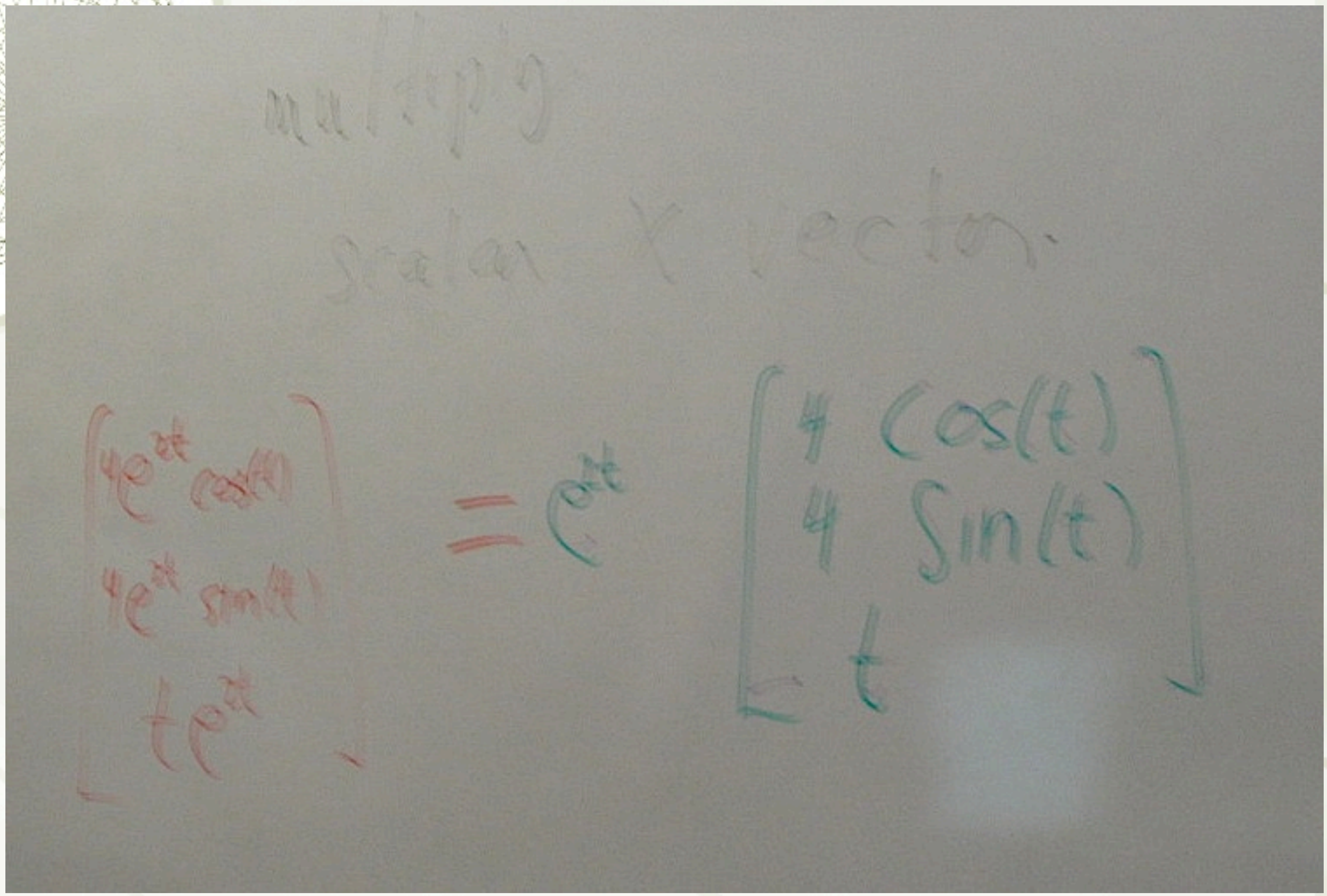
Any business to take care of?

- date Wed, Aug 20, 2008 at 2:26 PM
- subject Re: Math 2401 - homework question
- Dr. Harrell,
- Greetings! Thank you for all of the informative emails. I have a question in
- regards to the homework: are we to submit it during lectures on MW or during the
- recitation on T/Th? Thank you for your time. I look forward to hearing from you.
- Best regards,

ANSWER: Please submit your homework to the TA at recitation.


SALAS / HILLE / ETGEN
EIGHTH EDITION
CALCULUS
ONE AND SEVERAL VARIABLES

But first - A bit more Vector Boot Camp!


multiply:
scalar X vector:
4e^2t cos(t)
4e^2t sin(t)
te^2t
= e^2t
4 cos(t)
4 sin(t)
t

$$\frac{d}{dt}\begin{bmatrix} 4\cos(t) \\ 4\sin(t) \\ t \end{bmatrix} = \begin{bmatrix} -4\sin t \\ 4\cos t \\ 1 \end{bmatrix}$$

Why on earth would you differentiate a

- dot product
- cross product ?

Examples

- How fast is the angle between two vectors changing?

  $\cos\theta(t) = \mathbf{v}(t) \bullet \mathbf{w}(t)$

  (You'll need product and chain rule.)
- How fast is the angular momentum changing? $\mathbf{L} = \mathbf{r} \times \mathbf{p}$.

Why on earth would you integrate a vector function?

Examples

- Given velocity **v**(t) find position **x**(t).
- Power = **F•v** .

  Work is the integral of this. If, say, **v** is fixed, you can integrate **F** and then dot it with **v**.

The good news:

- The rules of vector calculus look just like the rules of scalar calculus

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- The rules of vector calculus look just like the rules of scalar calculus
- Integrals and derivs of $\alpha\,\mathbf{f}(t)$, $\mathbf{f}(t)+\mathbf{g}(t)$, etc.
- Also - because of this - you can always calculate component by component.

Calculus is built on the idea of a limit. What does a limit mean for vector functions?

- The limit

$$\lim_{t \to t_0} \mathbf{f}(t) = \mathbf{L}$$

means

$$\lim_{t \to t_0} \|\mathbf{f}(t) - \mathbf{L}\| = 0$$

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Some kind of scalar that depends on t

One of the great tricks of vector calculus:

- **If you can rewrite a vector problem in some way as a scalar problem, it becomes "kindergarten math."**

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- The limit

$$\lim_{t \to t_0} \mathbf{f}(t) = \mathbf{L}$$

means

$$\lim_{t \to t_0} \|\mathbf{f}(t) - \mathbf{L}\| = 0$$

$$\|\mathbf{f}(t) - \mathbf{L}\|^2 = (f_1 - L_1)^2 + (f_2 - L_2)^2 + (f_3 - L_3)^2$$

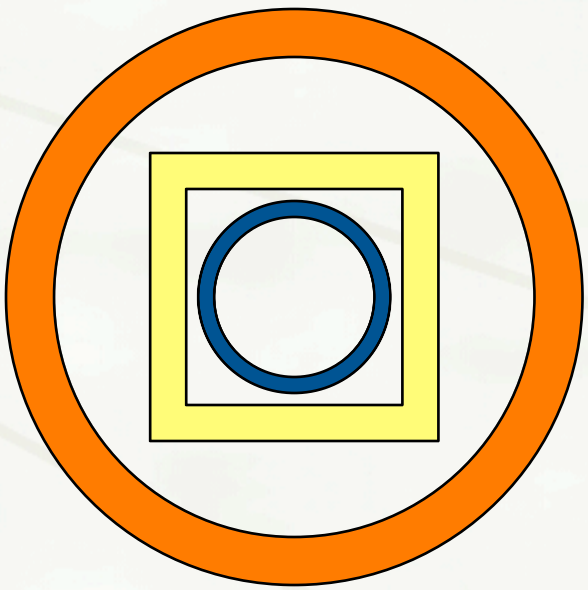
- So if the left side $\to 0$, each and every one of the contributions on the right $\to 0$ as well. And conversely.

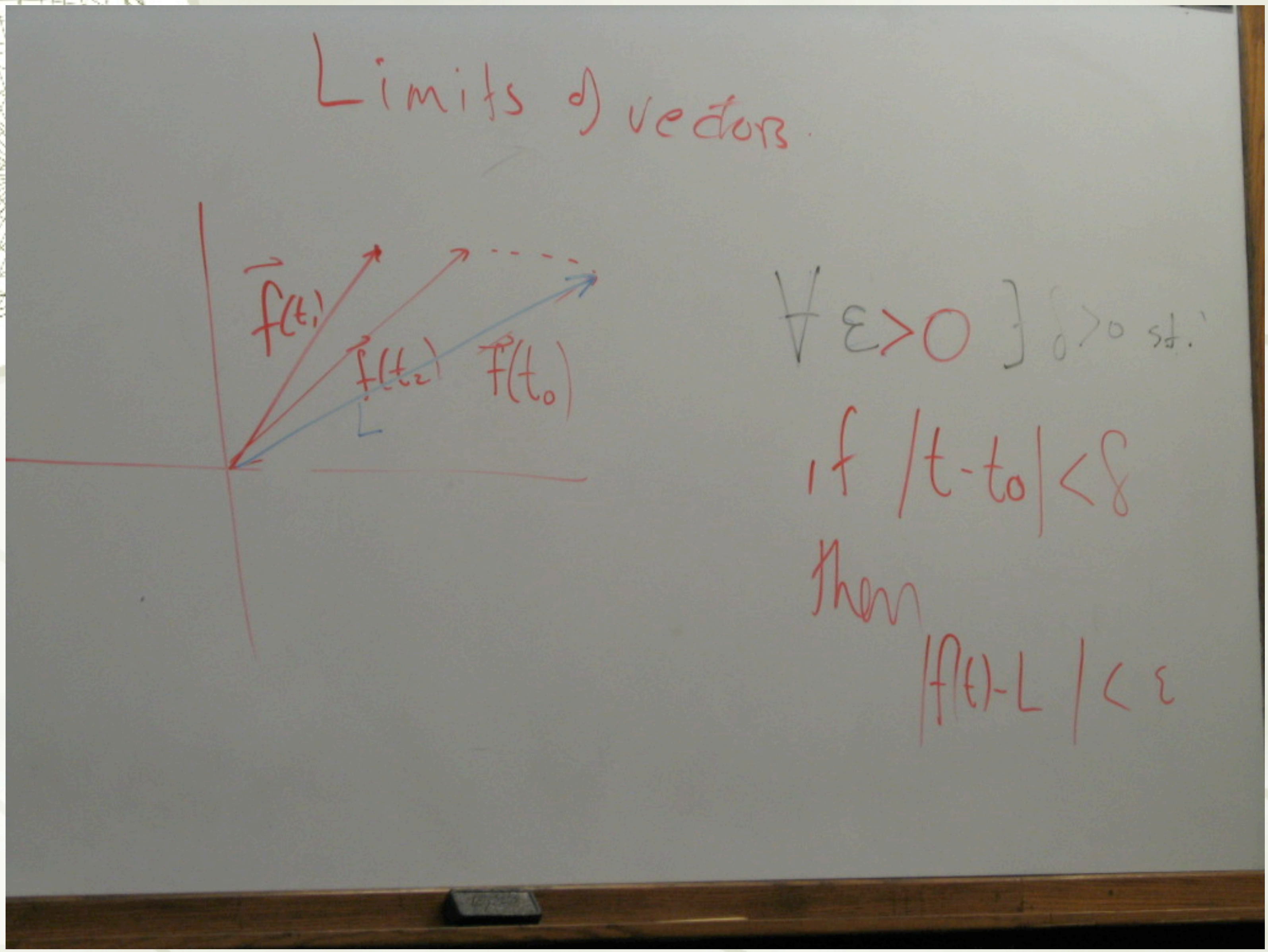
You can do calculus in terms of vectors or components.

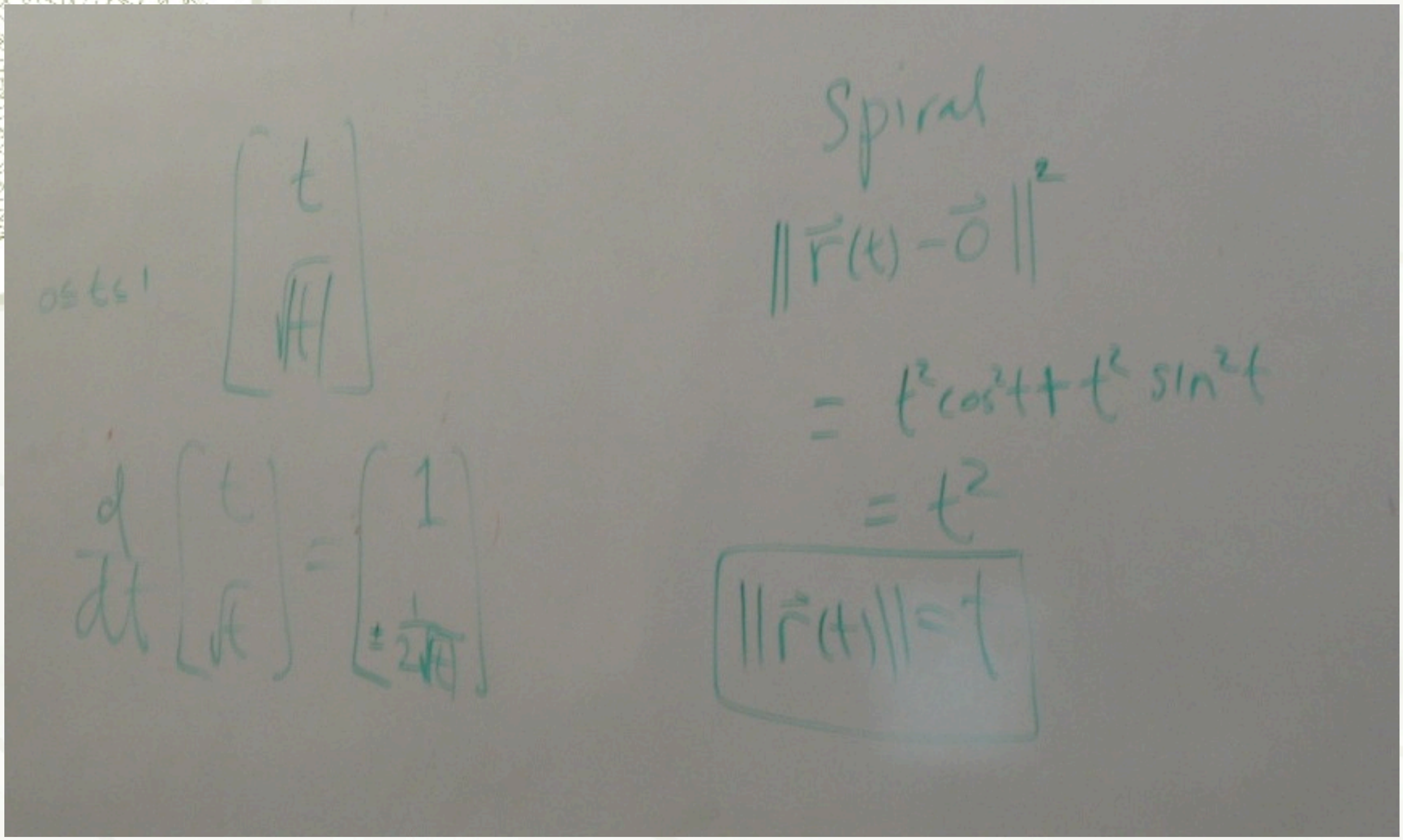
You choose.

You can think in terms of vectors or components.

A mystical picture:




Limits of vectors
f(t1)
f(t2)
f(t0)
L
∀ ε>0 ∃ δ>0 s.t.
if |t-t0|<δ
then
|f(t)-L| < ε


Spiral
||r(t) − 0||²
= t² cos² t + t² sin² t
= t²
||r(t)|| = t
0≤t≤1
t
√|t|
d/dt
t
√t
=
1
± 1/(2√|t|)

The derivative

$$\frac{d}{dt}\vec{V}(t) = \lim_{h\to 0} \frac{\vec{V}(t+h)-\vec{V}(t)}{h}$$

$$= \hat{i}\left(\lim_{h\to 0} \frac{V_1(t+h)-V_1(t)}{h}\right) + \hat{j}\left(\lim_{h\to 0} \frac{V_2(t+h)-V_2(t)}{h}\right)$$

etc.

The derivative

$$\frac{d}{dt}\vec{V}(t) = \lim_{h\to 0} \frac{\vec{V}(t+h)-\vec{V}(t)}{h}$$

Δt

What kind of animal?

A: vec.

$$= \hat{\imath}\left(\lim_{h\to 0}\frac{V_1(t+h)-V_1(t)}{h}\right)+\hat{\jmath}\left(\lim_{h\to 0}\frac{V_2(t+h)-V_2(t)}{h}\right)$$

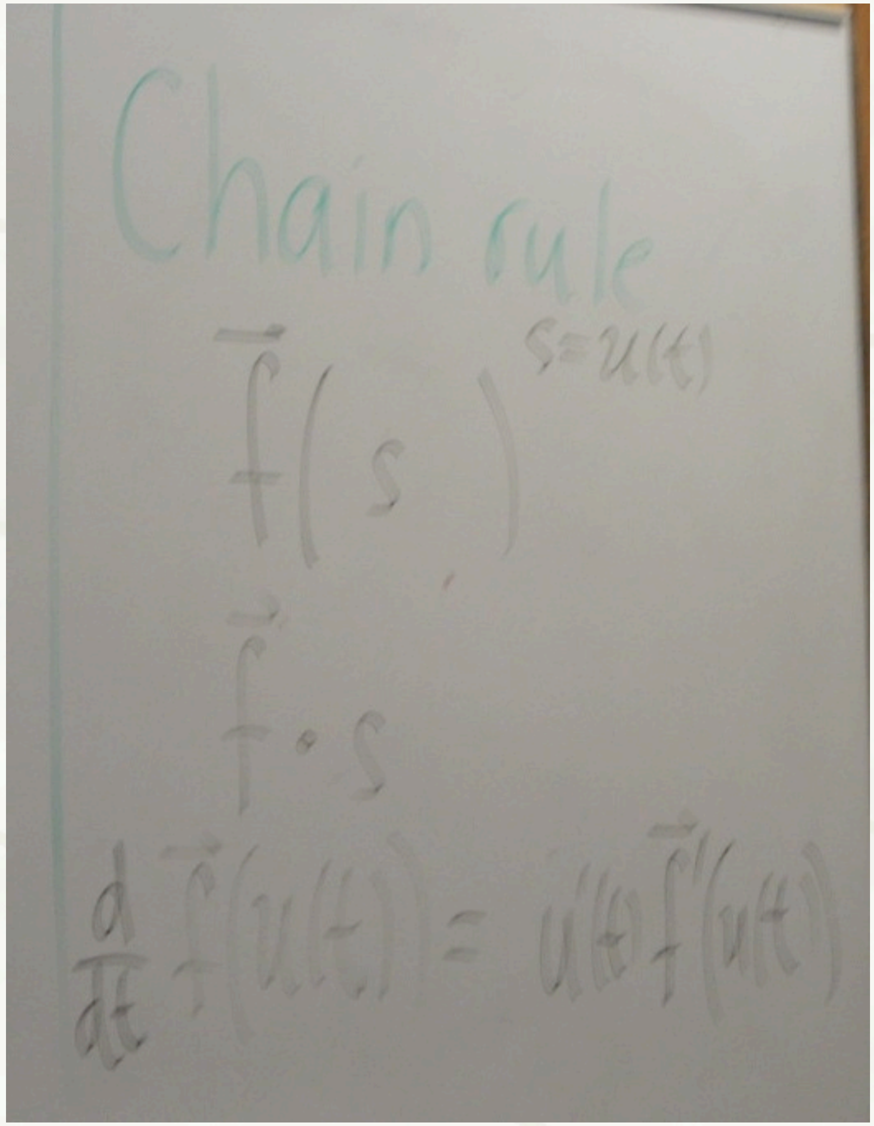
etc.

$$\vec{f}\cdot\vec{g} = f_1 g_1 + f_2 g_2$$

$$(\vec{f}\cdot\vec{g}) = f_1' g_1 + f_2' g_2$$

$$+ f_1 g_1' + f_2 g_2'$$

$$= (\vec{f}')\cdot g + \vec{f}\cdot\vec{g}'$$


Chain rule
s=u(t)
$\vec{f}(s)$
$\vec{f}$
$f \circ s$
$\frac{d}{dt}\vec{f}(u(t)) = u'(t)\vec{f}'(u(t))$

You can think in terms of vectors or components.

You choose.

- Some limit examples
  - spiral $x(t) = t \cos t$, $y(t) = t \sin t$
  - parabola $x(t) = t^2$, $y(t) = - t$

The good news:

- The rules of vector calculus look just like the rules of scalar calculus
- product rule(s)
  - $(u(t)\,\mathbf{f}(t))' = u'(t)\,\mathbf{f}(t) + u(t)\,\mathbf{f}'(t)$
  - $(\mathbf{f}(t)\bullet\mathbf{g}(t))' = \mathbf{f}'(t)\bullet\mathbf{g}(t) + \mathbf{f}(t)\bullet\mathbf{g}'(t)$
  - $(\mathbf{f}(t)\times\mathbf{g}(t))' = \mathbf{f}'(t)\times\mathbf{g}(t) + \mathbf{f}(t)\times\mathbf{g}'(t)$

The good news:

- The rules of vector calculus look just like the rules of scalar calculus
- chain rule
  - $(\mathbf{f}(u(t)))' = u'(t)\,\mathbf{f}'(u(t))$
- Example: If $u(t) = t^2$ and $f(x) = \sin(x)\mathbf{i} - 2x\,\mathbf{j}$, then $\mathbf{f}(u(t)) = \sin(t^2)\mathbf{i} - 2t^2\,\mathbf{j}$, and its derivative:
  - $2t\cos(t^2)\mathbf{i} - 4t\,\mathbf{j}$ is equal to
  - $2t(\cos(x)\mathbf{i} - 2\mathbf{j})$ when we substitute $x = t^2$.
- 2 ways to calculate: substitute and then differentiate, or chain rule

The good news:

- The rules of vector calculus look just like the rules of scalar calculus
- chain rule
  - $(f(u(t)))' = u'(t)\, f'(u(t))$

$\sin(t^2)\hat{\imath} - 2t^2\hat{\jmath}$

- Example: If $u(t) = t^2$ and $f(x) = \sin(x)\mathbf{i} - 2\,x\,\mathbf{j}$, then $\mathbf{f}(u(t)) = \sin(t^2)\mathbf{i} - 2\,t^2\,\mathbf{j}$, and its derivative:
  - $2\,t\,\cos(t^2)\mathbf{i} - 4\,t\,\mathbf{j}$ is equal to
  - $2\,t\,(\cos(x)\mathbf{i} - 2\mathbf{j})$ when we substitute $x = t^2$.

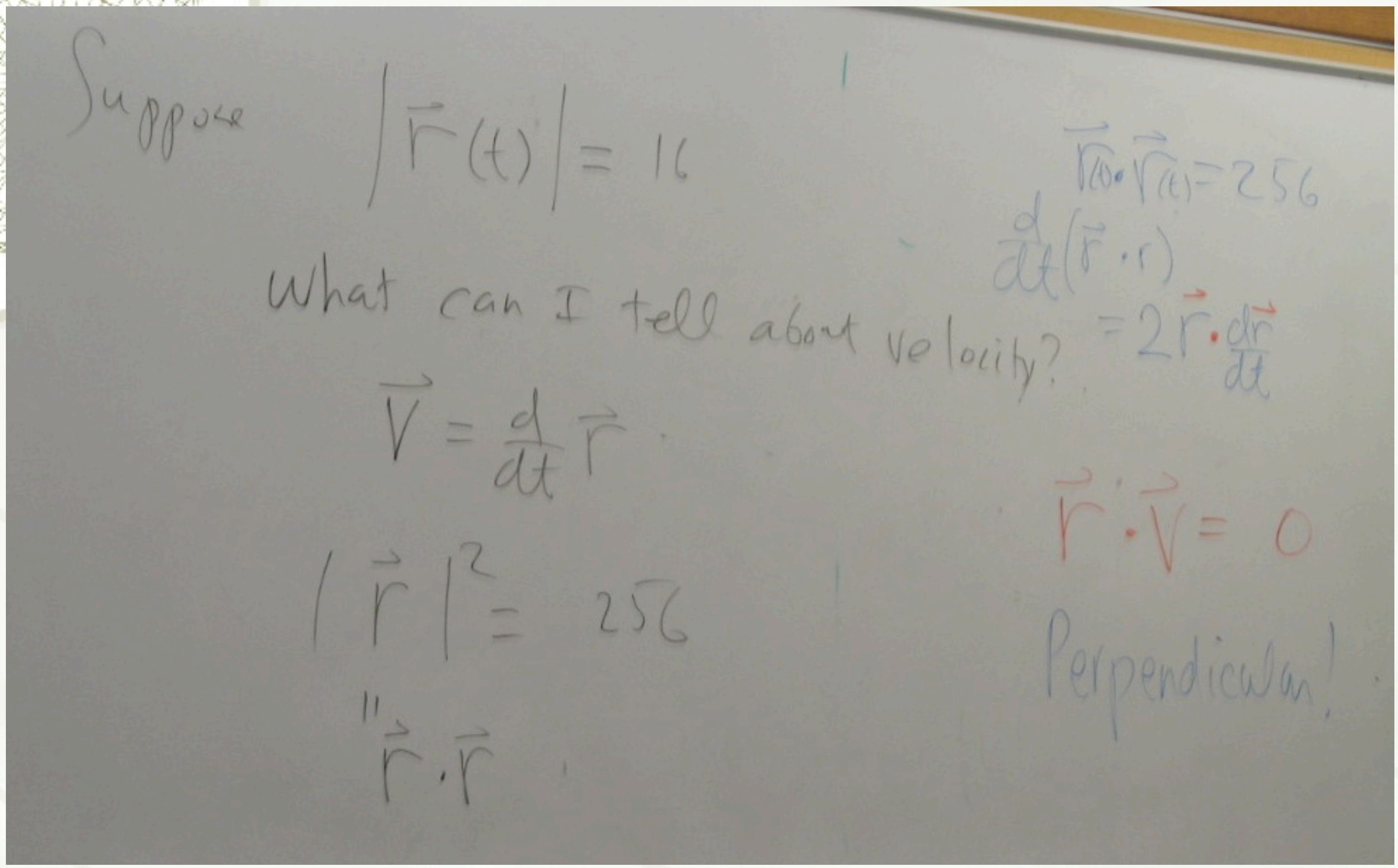
Some tricky stuff

Cross products. $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$. (In fact …..) What's right is right and what's left is left. Same for calculus.

Some tricky stuff

Suppose that the length of a vector **r**(t) is fixed. Then **r**(t) is always perpendicular to **r**'(t).

And vice versa.


Suppose $|\vec{r}(t)| = 16$
What can I tell about velocity?
$\vec{V} = \frac{d}{dt}\vec{r}$
$|\vec{r}|^2 = 256$
" $\vec{r}\cdot\vec{r}$
$\vec{r}(t)\cdot\vec{r}(t) = 256$
$\frac{d}{dt}(\vec{r}\cdot r)$
$= 2\vec{r}\cdot\frac{d\vec{r}}{dt}$
$\vec{r}\cdot\vec{V} = 0$
Perpendicular!

What about integrals of vectors? (Find position from velocity.)

How does it work?

Is there a +C? What kind of an animal is the +C?

The bottom line

Don't worry about the basic rules of calculus for vector functions. They are pretty much like the ones you know and love.

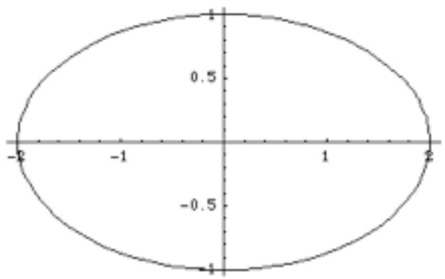
Now for the fun…Curves.

Some great curves and how to write them as parametrized curves

- circles and ellipses
- spirals
- helix
- Lissajous figures

Some Nice Curves

```
ParametricPlot[{2 Cos[t], Sin[t]}, {t, 0, 2 Pi}]
```

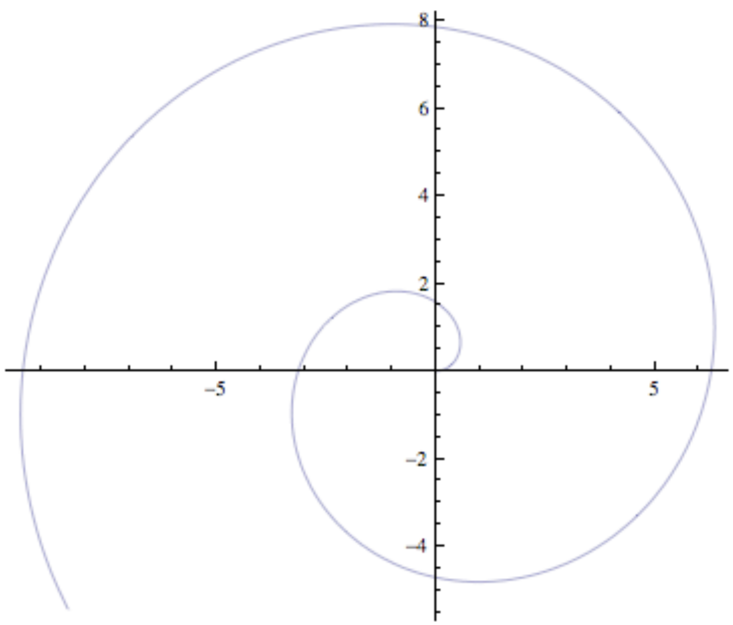



```
- Graphics -
```

```
Spiral[t_] := {t Cos[t], t Sin[t]}
```

```
Spiral3D[t_] := {t Cos[t], t Sin[t], 0}
```

```
ParametricPlot[Spiral[t], {t, 0, 10}]
```

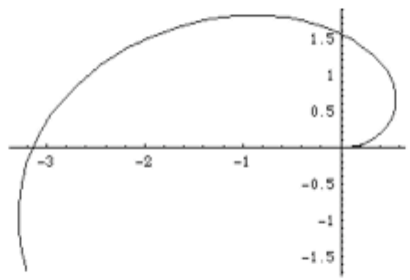


- What is we change the parameter t -> Sinh[s], for instance?

```
ParametricPlot[Spiral[Sinh[s]], {s, 0, 2}]
```

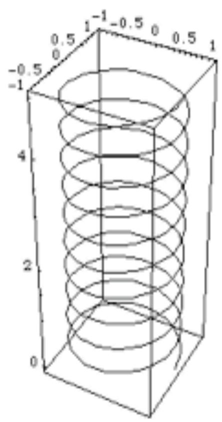
```
ParametricPlot3D[Spiral3D[t], {t, 0, 30}, ViewPoint → {5, 0, 4}]
```

```
ParametricPlot[Spiral[Sinh[s]], {s, 0, 2}]
```



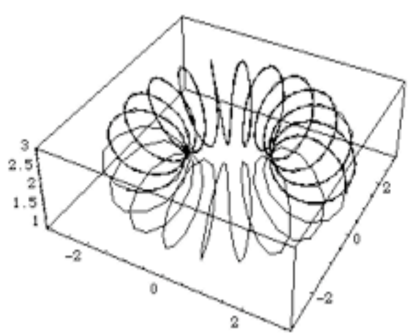
- Graphics -

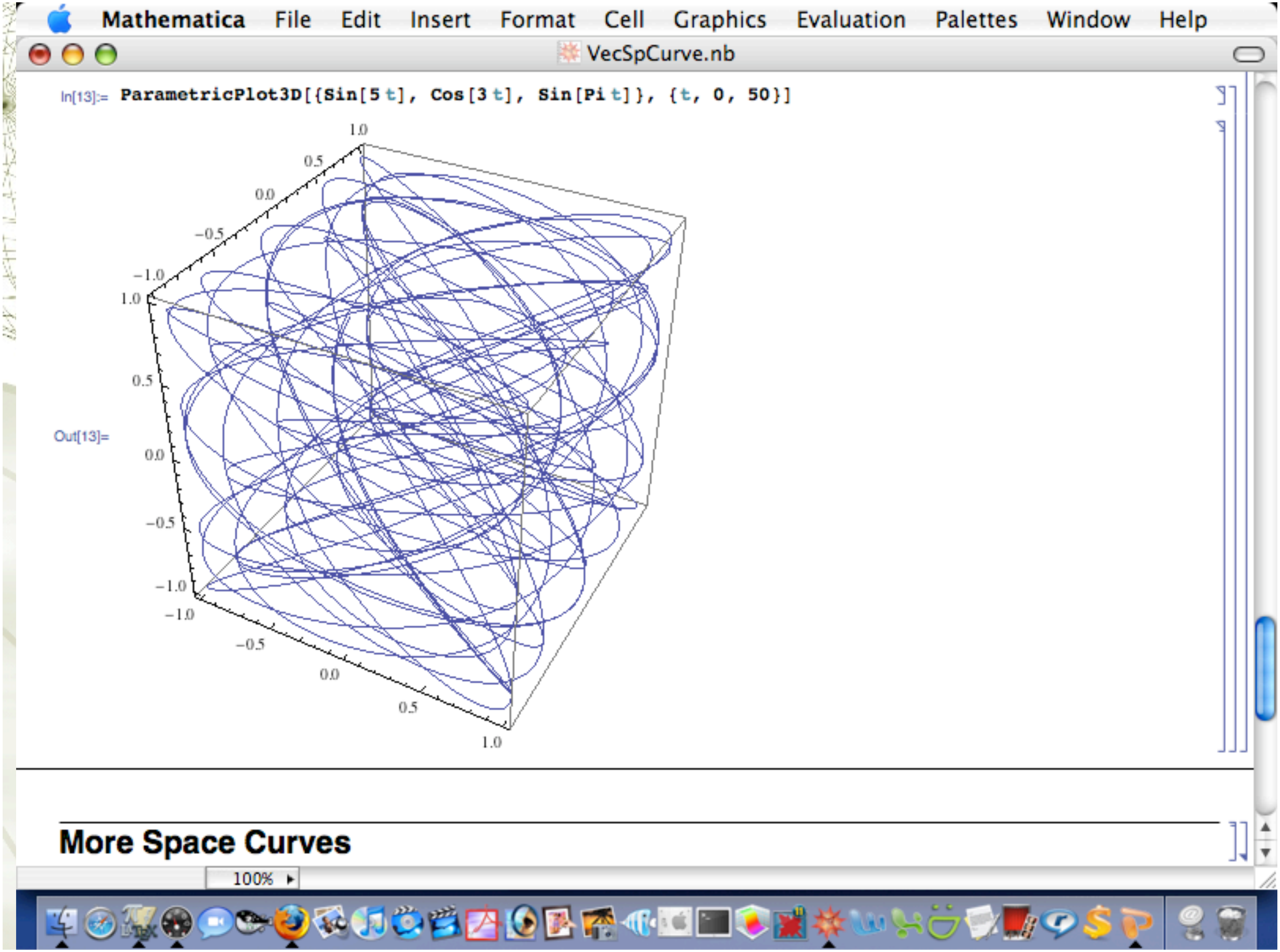
```
Helix[t_] := {Cos[4 Pi t], Sin[4 Pi t], t}

ParametricPlot3D[Helix[t], {t, 0, 5}, PlotPoints → 360]
```



- Graphics3D -

```
Solenoid[t_, r_, R_, w_] := {(R + r Cos[w t]) Cos[t], (R + r Cos[w t]) Sin[t], R + r Sin[w t]}

ParametricPlot3D[Solenoid[t, 1, 2, 20], {t, 0, 10}, PlotPoints → 360]
```




Mathematica File Edit Insert Format Cell Graphics Evaluation Palettes Window Help
VecSpCurve.nb
In[13]:= ParametricPlot3D[{Sin[5 t], Cos[3 t], Sin[Pi t]}, {t, 0, 50}]
Out[13]=
1.0
0.5
0.0
−0.5
−1.0
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0
More Space Curves
100%