Bewitching triple integrals!

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`Because Oct 31 = Dec 25 !`

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(Hint: There are 10 kinds of people in the world, those who understand binary and those who don’t. And the same for octal.)

Speaking of tests….

- The final exam is scheduled for Period 5, Tuesday, 9 December, 11:30-2:20.
  - Potential conflict with Physics 2212
  - Potential conflict with CS 1371 (Section E)

I have been in contact with Profs. Murray/ Greco about the conflict with Physics 2212. Write them by 10 Nov.

Write me about other conflicts ASAP!

Triple integrals

- Volume: $\int dx\, dy\, dz$

- Mass or other integrals:
  $\int \lambda(x,y,z) dx\, dy\, dz$

- Average:
  $(1/\mathrm{Vol}(\Omega)) \bullet \int \lambda(x,y,z) dx\, dy\, dz$

Triple integrals

- Integrals over boxes. Example. What is the total mass of a brick, $0 \le x \le 8, 0 \le y \le 4, 0 \le z \le 3$,
  - When the density is $10 + z \cos(\pi x)$?
  - When the density is $y z^2 e^{xyz}$?

- Brick, 0≤x≤8,0 ≤y≤4,0≤z≤3,
  - When the density is 10 + z cos(πx)?

$$\int_{\_\_\_\_\_\_}^{\_\_\_\_\_\_}\int_{\_\_\_\_\_\_}^{\_\_\_\_\_\_}\int_{\_\_\_\_\_\_}^{\_\_\_\_\_\_} \_\_\_\_\_\_\_\_\_\_\_\_\, d\_\_\_\_\_\_\, d\_\_\_\_\_\_\, d\_\_\_\_\_\_.$$

$$960 + \int_0^8 \cos \pi x \, dx \int_0^3 z \, dz \int_0^4 1 \, dy$$

$$= 960$$ with no work!

- Brick, 0≤x≤8,0 ≤y≤4,0≤z≤3,
  - When the density is y z^2 e^{xyz} ?

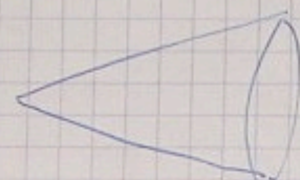
$$\int_0^3 \int_0^4 yz^2 \left(\int_0^8 e^{xyz}\, dx \right) dy\, dz$$

$$\int_0^3 \int_0^4 \left(yz^2 \frac{1}{yz} e^{xyz} \Big|_0^8 \right) dy\, dz$$

$$= \int_0^3 z \left(\int_0^4 \left(e^{8yz} - e^{0}\right) dy \right) dz$$

$$= \int_0^3 z\left(\frac{1}{8z}\left(e^{32z}-1\right)-4\right) dz$$

Volume of the cone $0 \le y \le 2,\ x^2+z^2 \le y^2$



$$\text{Vol} = \int_0^2 \int_{-y}^{y} \int_{-\sqrt{y^2-x^2}}^{+\sqrt{y^2-x^2}} 1 \; dz\,dx\,dy = \int_0^2 \int_{-y}^{y} 2\sqrt{y^2-x^2}\,dx\,dy$$

$$= \int_0^2 \pi y^2\,dy = \frac{8\pi}{3}$$

Centroid:

$$y_m = \frac{3}{8\pi} \int_0^2 \int_{-y}^{y} \int_{-\sqrt{y^2-x^2}}^{\sqrt{y^2-x^2}} y\; dz\,dx\,dy = \frac{3}{8\pi} \int_0^2 y \cdot \pi y^2\,dy$$

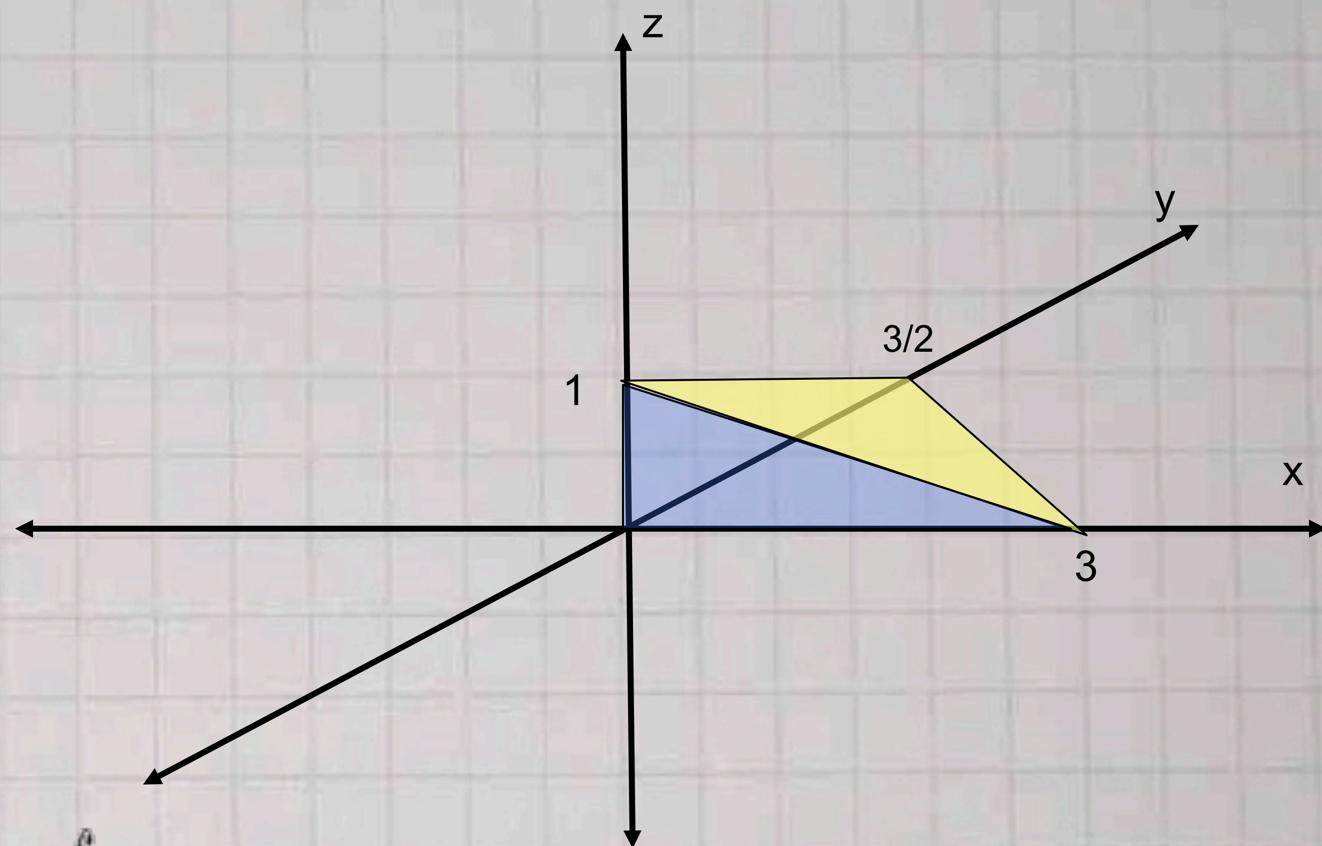
$$= \frac{3}{32} 2^4 = \frac{3}{2}$$

$\frac{1}{4}$ of the way from the fat end to the tip.

Triple integrals

- Examples:
  - Cone $y^2 = x^2 + z^2$ Volume? Centroid?
  - Wedge $0 \le x, y, z$, $x + 2y + 3z \le 3$.
  - Integral of $\sin x \sin y \sin z$ with Ω bounded by $z = y$, $z = 0$, $x = 0$, $x = \pi/2$, $y = \pi$.
  - Integral of $y^2 x^2 z$ with $0 \le z \le x^2 - y^2$, $0 \le x \le 1$.

- Volume of wedge $0 \le x,y,z$, $x + 2y + 3z \le 3$.



$$\int_{\_\_\_\_\_\_}^{\_\_\_\_\_\_} \int_{\_\_\_\_\_\_}^{\_\_\_\_\_\_} \int_{\_\_\_\_\_\_}^{\_\_\_\_\_\_} \_\_\_\_\_\_\_\_\_\_\_\_ \, d\_\_\_\_\_\_ \, d\_\_\_\_\_\_ \, d\_\_\_\_\_\_.$$

2nd int.

$$\int_0^{3-3z} \frac{1}{2}(3 - x - 3z)\,dx$$

$$= \frac{1}{2}(3-3z)(3-3z) - \frac{1}{2}\int_0^{3-3z} x\,dx$$

$$= \frac{1}{2}(3-3z)^2 - \frac{1}{4}(3-3z)^2$$

$$= \frac{1}{4}(3-3z)^2 = \frac{9}{4}(1-z)^2$$

3rd int.

$$\frac{9}{4}\int_0^1 (1-z)^2\,dz$$

$$= \frac{3}{4} \quad (?)$$

Triple integrals

- Example:

  - Integral of sin x sin y sin z with Ω bounded by
    z = y, z = 0, x = 0, x = π/2, y = π.

$$\int_{\_\_\_\_\_\_\_}^{\_\_\_\_\_\_\_} \int_{\_\_\_\_\_\_\_}^{\_\_\_\_\_\_\_} \int_{\_\_\_\_\_\_\_}^{\_\_\_\_\_\_\_} sinx siny sinz \, d\_\_\_\_\_\_\_ d\_\_\_\_\_\_\_ d\_\_\_\_\_\_\_.$$

Triple integrals

- Example:

- Integral of sin x sin y sin z with Ω bounded by
  z = y, z = 0, x = 0, x = π/2, y = π.

$$\int_{0}^{\pi}\int_{z}^{\pi}\int_{0}^{\pi/2} \sin x \sin y \sin z \, dx\, dy\, dz.$$

$$2^{nd}: \quad \int_{z}^{\pi} \sin y \, dy$$

$$= \int_{0}^{\pi} \sin z \left(1 + \cos z\right) dz$$

Another fun game

- Limits of integration. What is the region of integration in

$$\int_0^4 \int_0^{4-x} \int_0^{4-x-y} \_\_\_\_\_\_\_\_\_\_\_\_ d\_\_\_\_\_\_ d\_\_\_\_\_\_ d\_\_\_\_\_\_.$$

- How does it look if we integrate in a different order?

The switcheroo is back!

$z \le 4 - x - 0$

$y \le 4 - x - z$

$0 \le y \le 4 - x$

$0 \le z \le 4 - x - y$

$0 \le x \le 4$

$$\int_0^4 \int_0^{4-x} \int_0^{4-x-y} \_\_\_\_\_\_\_\_ d\_\_\_\_ d\_\_\_\_ d\_\_\_\_.$$

$$\int_0^4 \int_0^{4-x} \left[\int_0^{4-x-z} \_\_\_\_\_\_\_\_ dy \right] dz \, dx.$$

Another fun game

- Limits of integration. What is the region of integration in

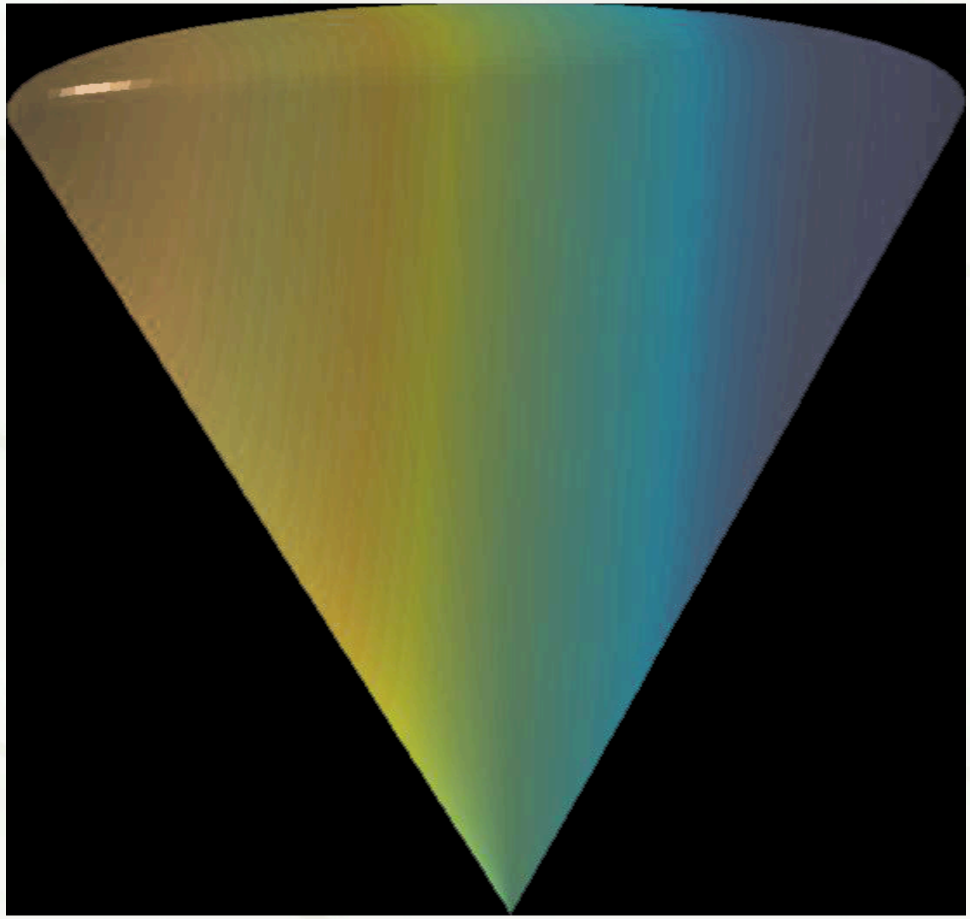
$$\int_0^4 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} \underline{\hspace{3cm}}\, d\underline{\hspace{1.5cm}}\, d\underline{\hspace{1.5cm}}\, d\underline{\hspace{1.5cm}}.$$

- How does it look if we integrate in a different order?

Coordinate systems for grown-ups

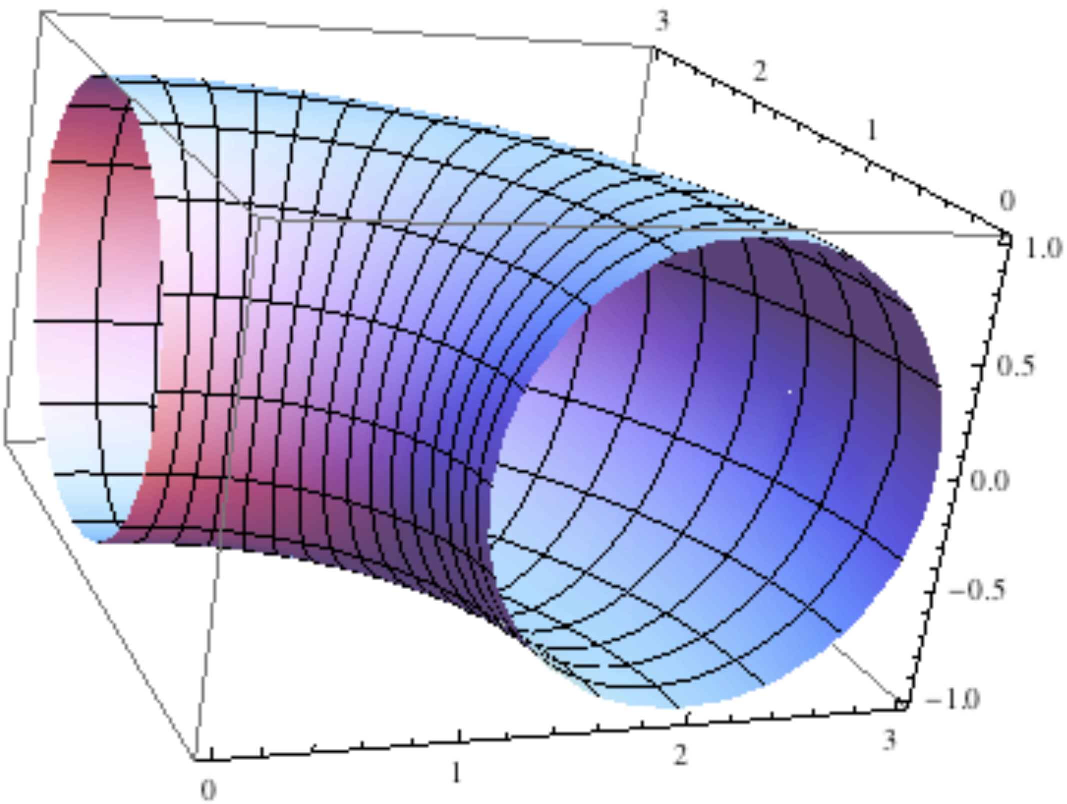
- Cylindrical = polar plus z

- Spherical = geographic coordinates plus radius

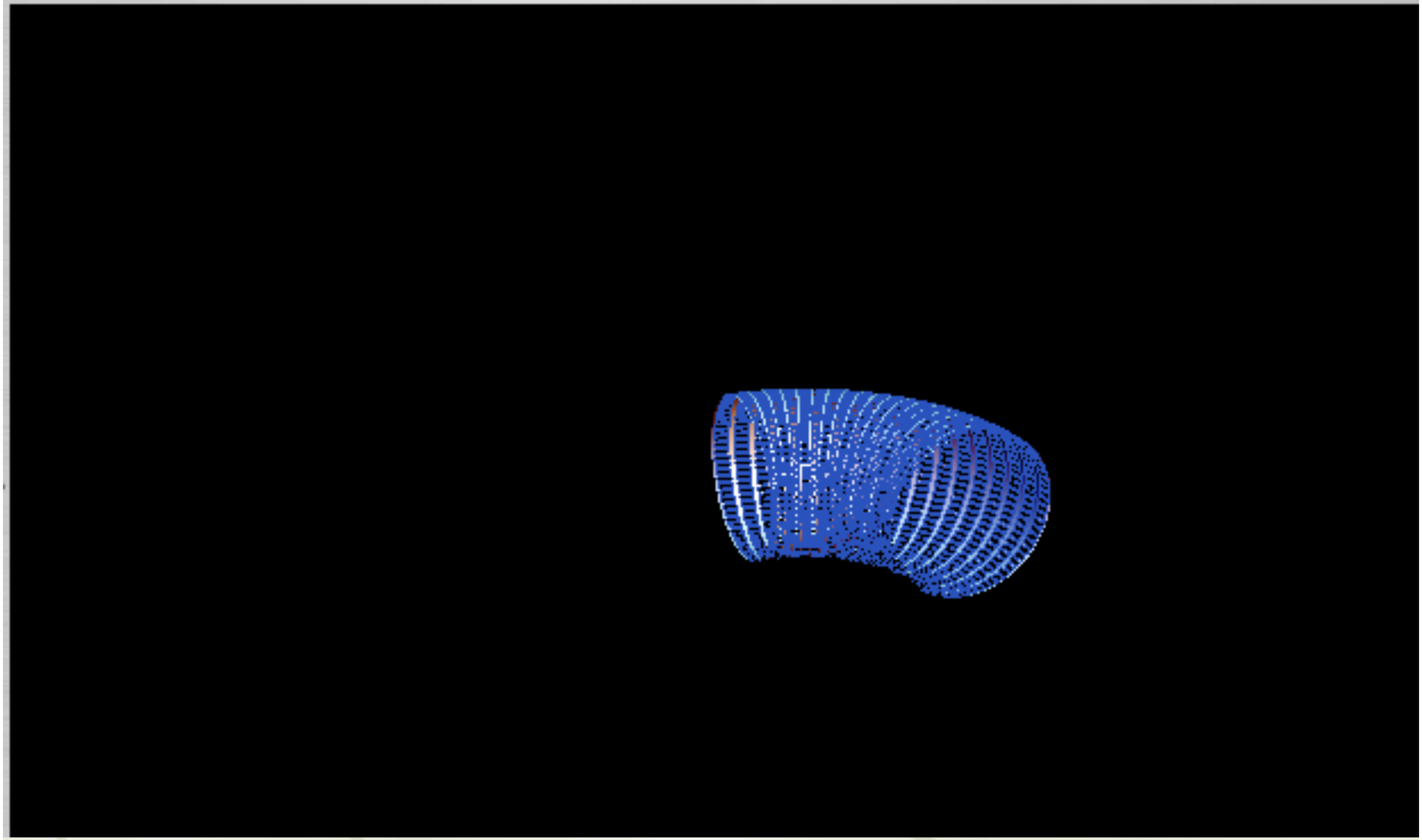


In[3]:=
```
ParametricPlot3D[{(2 + Cos[t]) Cos[s], (2 + Cos[t]) Sin[s], Sin[t]},
  {s, 0, Pi / 2}, {t, 0, 2 Pi}]
```

Out[3]=



$$\begin{bmatrix} r_0 \\ \phi \\ z \end{bmatrix} = \begin{bmatrix} 2+\cos u \\ t \\ \sin u \end{bmatrix}, t=0\ldots\frac{\pi}{2}, u=0\ldots2\pi$$



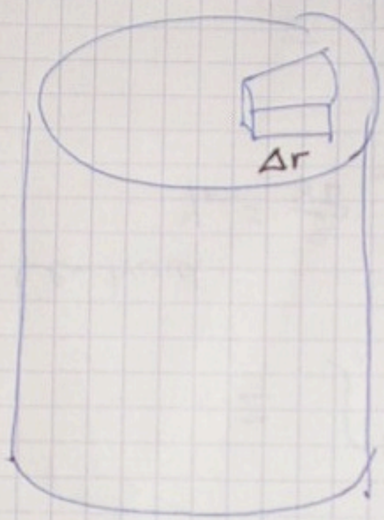
Coordinate systems for grown-ups

- Cylindrical = polar plus z
  - r = distance from vertical axis
  - θ = angle
  - z = height

Coordinate systems for grown-ups

- Cylindrical to Cartesian:
  - $x = r \cos \theta$
  - $y = r \sin \theta$
  - $z = z$

How big is ΔV ?



in horizontal plane

$$dA = r\,dr\,d\theta$$

height $= dz$

$$dV = r\,dr\,d\theta\,dz$$

Cylindrical examples

- Volume of sphere
- Volume of 1/4 torus (doughnut)
- Integral of $z (x^2 + y^2 + z^2)^{1/2}$, where
  - $x,y > 0$ and
  - z is between $(x^2 + y^2)^{1/2}$ and $(1-x^2-y^2)^{1/2}$
- *What does this region look like?*

Cylindrical examples

- Volume of sphere
- Volume of 1/4 torus (doughnut)
- Integral of $z\,(x^2 + y^2 + z^2)^{1/2}$, where
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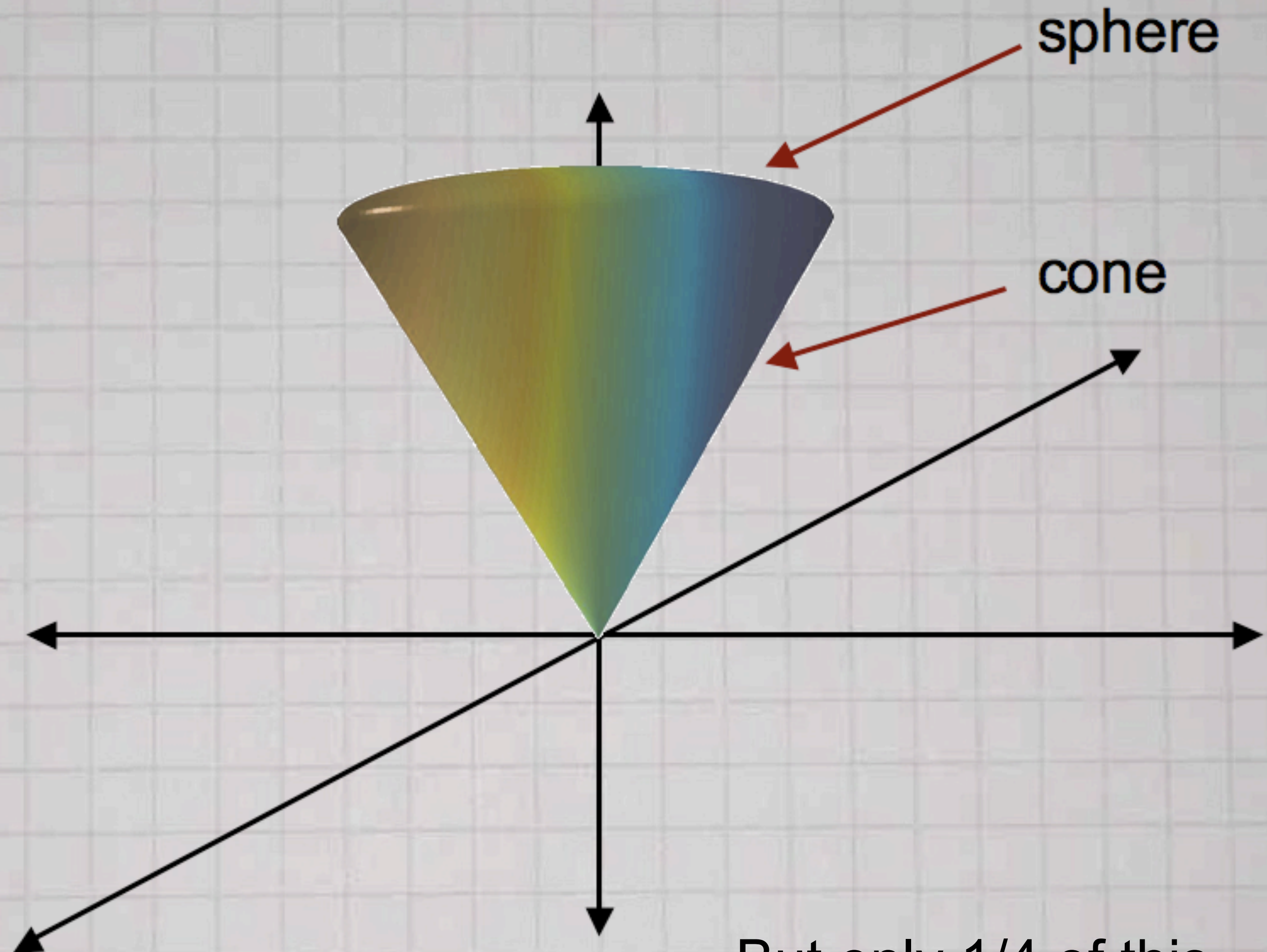
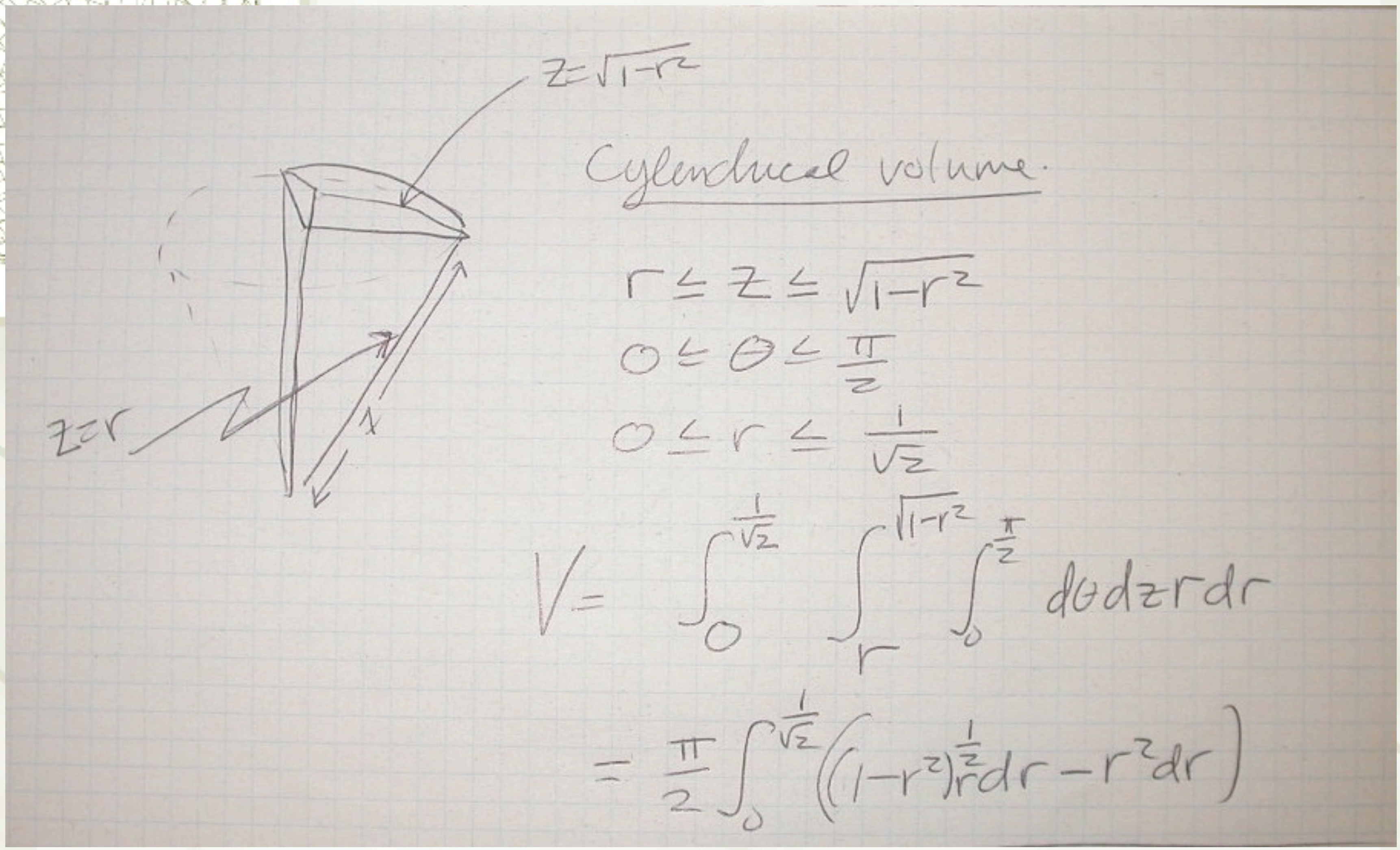
- *What does this region look like?*



But only 1/4 of this.

Cylindrical examples



$$V = \frac{\pi}{2}\left(-\frac{1}{3}(1-r^2)^{\frac{3}{2}} - \frac{1}{3}r^3 \Big|_0^{\frac{1}{\sqrt{2}}} \right)$$

$$= \frac{\pi}{6}\left(1 - \frac{1}{2^{3/2}} - \frac{1}{2^{3/2}}\right) = \boxed{\frac{\pi}{6}\left(1 - \frac{1}{\sqrt{2}}\right)}$$

Coordinate systems for grown-ups

- Spherical = geographic plus ρ
  - ρ = distance from origin
  - θ = polar angle in xy plane = longitude
  - ϕ = angle from pole, “colatitude”

The End