Coordinates all around

Coordinate systems for grown-ups

- Cylindrical = polar plus z
- Spherical = geographic coordinates plus radius

Coordinate systems for grown-ups

- Cylindrical = polar plus z
  - r = distance from vertical axis, $0 \le r$
  - θ = angle, any range of length 2π
  - z = height, $-\infty < z < \infty$

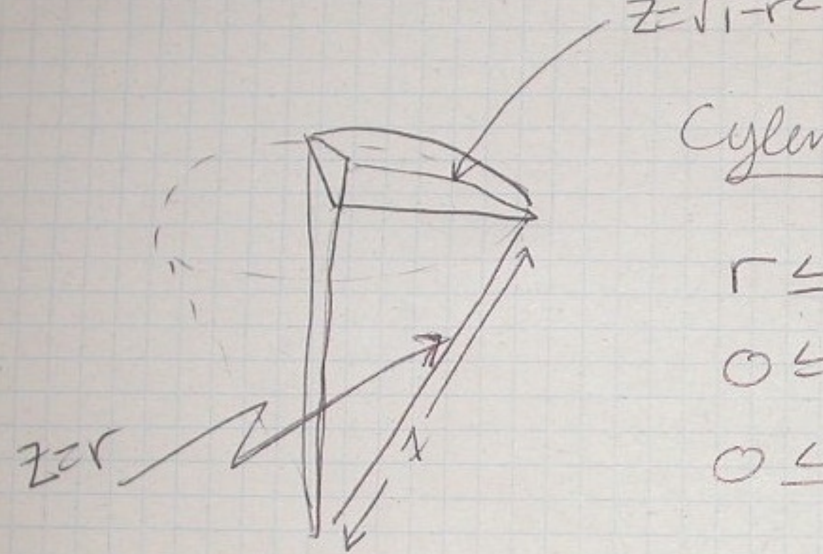
Coordinate systems for grown-ups

- Cylindrical to Cartesian:
  - $x = r \cos \theta$
  - $y = r \sin \theta$
  - $z = z$

Coordinate systems for grown-ups

- Cartesian to Cylindrical :
  - $r = (x^2 + y^2)^{1/2}$
  - $\theta = \arctan(y/x)$
  - $z = z$

Cylindrical examples



Cylindrical volume

$$r \le z \le \sqrt{1-r^2}$$

$$0 \le \theta \le \frac{\pi}{2}$$

$$0 \le r \le \frac{1}{\sqrt{2}}$$

$$V = \int_0^{\frac{1}{\sqrt{2}}} \int_r^{\sqrt{1-r^2}} \int_0^{\frac{\pi}{2}} d\theta\, dz\, r\, dr$$

$$= \frac{\pi}{2} \int_0^{\frac{1}{\sqrt{2}}} \left((1-r^2)^{\frac{1}{2}} r\, dr - r^2 dr \right)$$

$$V = \frac{\pi}{2}\left(-\frac{1}{3}(1-r^2)^{\frac{3}{2}} - \frac{1}{3}r^3 \Big|_0^{\frac{1}{\sqrt{2}}} \right)$$

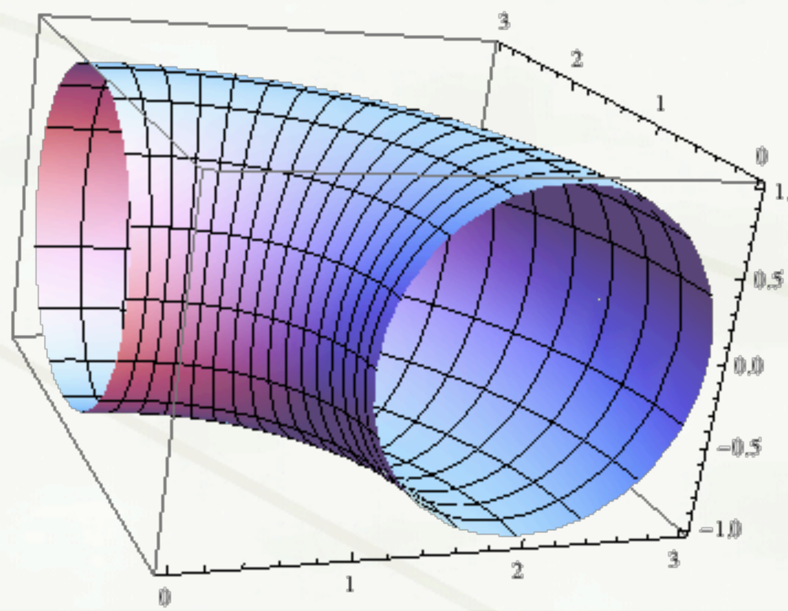
$$= \frac{\pi}{6}\left(1 - \frac{1}{2^{3/2}} - \frac{1}{2^{3/2}}\right) = \boxed{\frac{\pi}{6}\left(1 - \frac{1}{\sqrt{2}}\right)}$$

Cylindrical examples

- Volume of 1/4 torus (doughnut)

```
In[8]:= ParametricPlot3D[{(2 + Cos[t]) Cos[s], (2 + Cos[t]) Sin[s], Sin[t]},
   {s, 0, Pi / 2}, {t, 0, 2 Pi}]
```

Out[8]=



Cylindrical examples

- Volume of 1/4 torus (doughnut)

$$= \frac{\pi}{2} \int_{-1}^{1} \frac{1}{2} r^2 \Big|_{2-\sqrt{1-z^2}}^{2+\sqrt{1-z^2}} dz$$

$$= \frac{\pi}{4} \int_{-1}^{1} \left(\left(4 + 4\sqrt{1-z^2} + (1-z^2)\right) - \left(4 - 4\sqrt{1-z^2} + (1-z^2)\right) \right) dz$$

$$V_\theta = \int_{-1}^{1} \int_{2-\sqrt{1-z^2}}^{2+\sqrt{1-z^2}} \int_0^{\pi/2} d\theta \, r\,dr\,dz \qquad 0 \le \theta \le \pi/2$$

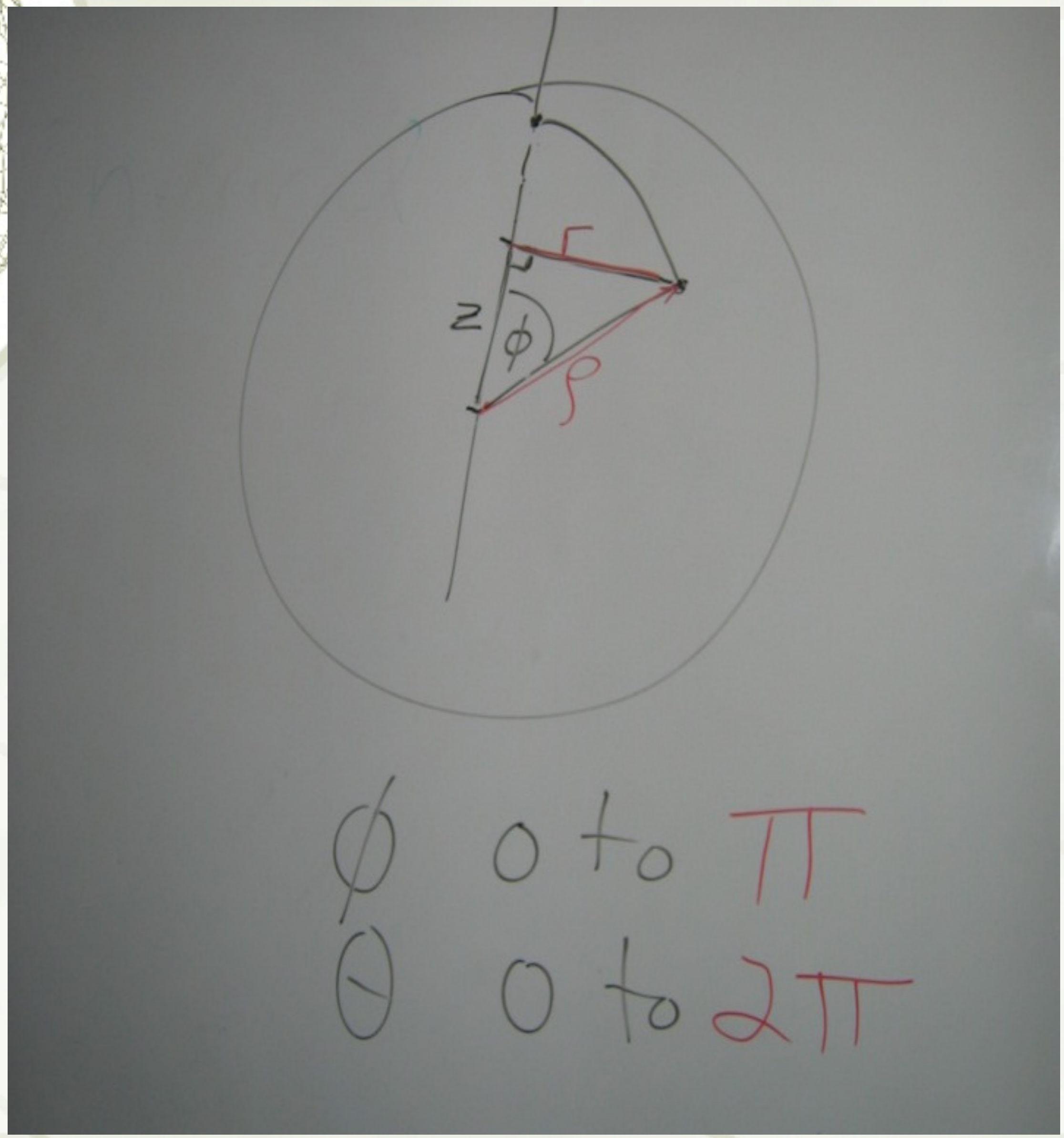
$$= \frac{8}{4}\pi \int_{-1}^{1} \sqrt{1-z^2}\, dz$$

$$= 2 \cdot \pi \cdot \frac{\pi}{2}$$

$$= \pi^2$$

Coordinate systems for grown-ups

- Spherical = geographic plus ρ
  - ρ = distance from origin
  - θ = polar angle in xy plane = longitude
  - ϕ = angle from pole, "colatitude"


r
z
ϕ
ρ
ϕ 0 to π
Θ 0 to 2π

Coordinate systems for grown-ups

- Cartesian to spherical:
  - $\rho = (x^2 + y^2 + z^2)^{1/2}$
  - $\tan \theta = y/x;$
    - or $\cot \theta = x/y;$
    - or $\cos \theta = x/(x^2 + y^2)^{1/2}$
  - $\cos \phi = z/(x^2 + y^2 + z^2)^{1/2}$

Coordinate systems for grown-ups

- Spherical to cylindrical:
  - $r = \rho \sin \phi$
  - $\theta = \theta$
  - $z = \rho \cos \phi$

Coordinate systems for grown-ups

- Spherical to Cartesian:
  - $x = \rho \sin \phi \cos \theta$
  - $y = \rho \sin \phi \sin \theta$
  - $z = \rho \cos \phi$

Nice application

- How far is it by the shortest air route from Atlanta to Moscow?
  - Atlanta
    - latitude=33.640200544698 °
    - longitude= -84.418068587706 °
  - Moscow
    - latitude=55.57751294796784 °
    - Longitude=37.76779917676628 °

Nice application

- How far is it by the shortest air route from Atlanta to Moscow?

  Note: ρ = 6378 km

  - Atlanta
    - ϕ = colatitude= .98366 radians
      - sin ϕ = .8325, cos ϕ = .5540
    - θ = longitude= -1.4734 radians
      - sin θ = -.9953, cos θ = .0973

Nice application

- How far is it by the shortest air route from Atlanta to Moscow?

  Note: ρ = 6378 km

  - Moscow
    - ϕ = colatitude= .60079 radians
      - sin ϕ = .5653, cos ϕ = .8249
    - θ = longitude= .65917 radians
      - sin θ = .6125, cos θ = .7905

Nice application

- How far is it by the shortest air route from Atlanta to Moscow?

  Note: ρ = 6378 km

  - Atlanta
    - $x = \rho\ (.8325)(.0973) = .0810$
    - $y = \rho\ (.8325)(-.9953) = -.8286$
    - $z = \rho\ (.5540)$

Nice application

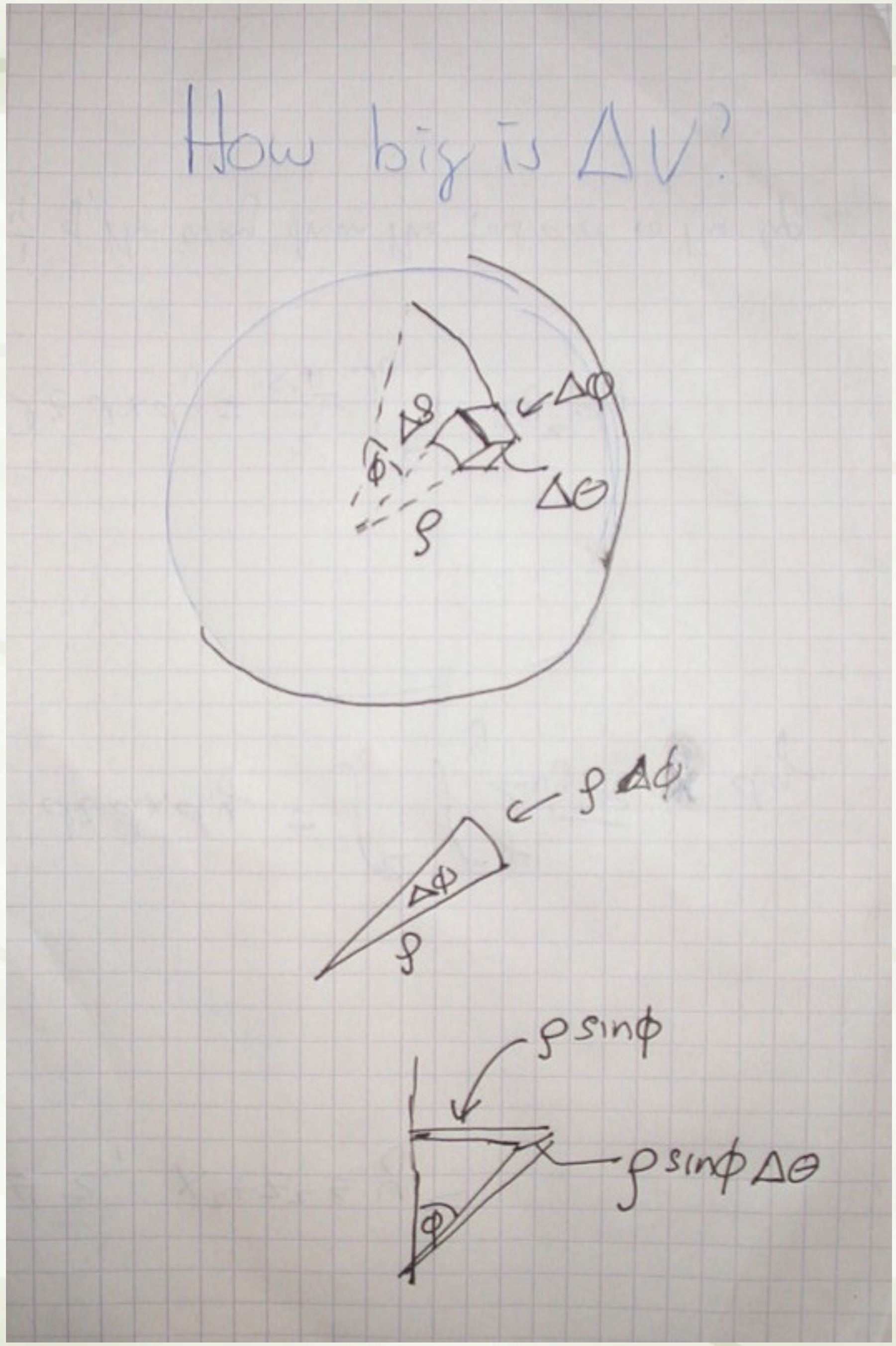
- How far is it by the shortest air route from Atlanta to Moscow?

  Note: ρ = 6378 km

  - Moscow
    - $x = \rho\,(.5653)(.7905) = .4469$
    - $y = \rho\,(.5653)(.6125) = .3462$
    - $z = \rho\,(.8249)$

  $\text{Cos}(\alpha) = .4469*.0810 + .3462*(-.8286) + .8249*.5540$

  α = 1.36297, dist = 6378 α = 8693 km.


How big is ΔV?
Δρ
ΔΦ
φ
Δθ
ρ
ρΔφ
Δφ
ρ
ρ sinφ
ρ sinφ Δθ
φ

$\Delta\rho$ depth

$\rho\,\Delta\phi$ N-S edge

$\rho \sin\phi\,\Delta\theta$ E-W dist $(r\Delta\theta)$

$$\Delta V \cong \rho^2\,\Delta\phi \sin\phi\,\Delta\theta\,\Delta\rho$$

Sph $dV = \rho^2 \sin\phi\; d\rho\, d\theta\, d\phi$

Cy $dV = r\; dr\, d\theta\, dz$

Cylindrical examples

- Volume of ¼ cone with cap
- Volume of sphere
- Volume of a sliced sphere

Spherical volume

$0 \le \theta \le \frac{\pi}{2}$

$0 \le \varphi \le \pi/4$

$0 \le \rho \le 1$

no mixed limits!

$$\int_0^{\pi/2} \int_0^{\pi/4} \int_0^1 \rho^2 \, d\rho \, \sin\varphi \, d\varphi \, d\theta$$

$$= \frac{1}{3}\left[-\cos\left(\frac{\pi}{4}\right)+1\right] \cdot \frac{\pi}{2} = \frac{\pi}{6}\left(1-\frac{1}{\sqrt{2}}\right)$$

Prof. H's special spherical tips

- Remember, θ runs from 0 to 2 π, but ϕ runs only from 0 to π.
- Total "steradians" on the sphere = 4π = the complete integral of $\sin\phi \, d\phi \, d\theta$
- Very often you want to use the variable $w = \cos\phi$, instead of ϕ. This variable runs from -1 to 1 and the volume element $dV = \rho^2 \, d\rho \, dw \, d\theta$.

  $dw = -\sin\phi \, d\phi$

Another one of the great integrals

- How can we design a spaceship that travels according to plan in the solar system, responding to gravitational forces from every particle in the universe, when we don't know the mass density λ inside the sun or the planets? All we know is that λ depends only on ρ, not θ or ϕ.

Another one of the great integrals

- Show that a spaceship can treat celestial bodies as if they are concentrated at a point. Until it crashes into the surface.

Another one of the great integrals

Another one of the great integrals

- Newton's theorem: If the mass density depends only on ρ, and becomes 0 above a certain value of ρ, then the gravitational potential outside the planet is the same as if all mass is concentrated at the origin, or …

Another one of the great integrals

$$\int_0^R \int_0^\pi \int_0^{2\pi} \frac{\lambda(\rho)}{|\mathbf{a}-\mathbf{r}|}\rho^2 \sin(\phi)\, d\theta d\phi d\rho = \frac{M}{|\mathbf{a}|},$$

$$M := \int_0^R \int_0^\pi \int_0^{2\pi} \lambda(\rho)\rho^2 \sin(\phi)\, d\theta d\phi d\rho.$$

Tune in Wednesday for the exciting dénouement of Newton's integral!