From our last episode....

- How can we design a spaceship that travels according to plan in the solar system, responding to gravitational forces from every particle in the universe, when we don't know the mass density λ inside the sun or the planets? All we know is that λ depends only on ρ, not θ or ϕ.

Another one of the great integrals

- Newton's theorem: If the mass density depends only on ρ, and becomes 0 above a certain value of ρ, then the gravitational potential outside the planet is the same as if all mass is concentrated at the origin:

Another one of the great integrals

- If the force follows the inverse-square law,

$$\mathbf{F}_{pt} = -\frac{G_m q Q}{|\mathbf{r}|^2} \cdot \frac{\mathbf{r}}{|\mathbf{r}|}$$

then

$$\mathbf{F}_{pt} = \nabla \frac{G_m q Q}{|\mathbf{r}|}$$

so the “gravitational potential” will be proportional to the integral of 1/ |**r**| times the mass density.

Another one of the great integrals

$$\int_0^R \int_0^\pi \int_0^{2\pi} \frac{\lambda(\rho)}{|\mathbf{a}-\mathbf{r}|}\rho^2 \sin(\phi) d\theta d\phi d\rho = \frac{M}{|\mathbf{a}|},$$

$$M := \int_0^R \int_0^\pi \int_0^{2\pi} \lambda(\rho)\rho^2 \sin(\phi) d\theta d\phi d\rho.$$

$dx\,dy$

$\vec{a} = a\hat{k}$

$=$

Spherical coords

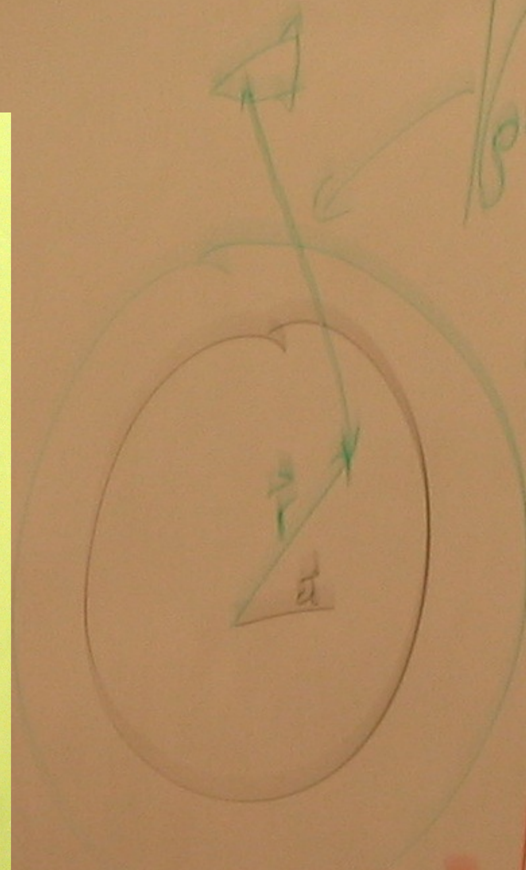
$\rho_a = a$

$\phi_a = 0$

$\theta_a = \text{undef.}$

Integration variables

$$\vec{r} = \begin{pmatrix} \rho \sin\phi \cos\theta \\ \rho \sin\phi \sin\theta \\ \rho \cos\phi \end{pmatrix}$$



$$\left\| \left(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ a-\rho\cos\phi \right) \right\|$$

$$= \sqrt{(a-\rho\cos\phi)^2 + \rho^2\sin^2\phi}$$

$$= \sqrt{a^2 - 2a\rho\cos\phi + \rho^2}$$

$$I = \int_0^R \int_0^\pi \int_0^{2\pi} \frac{\lambda(\rho)}{\sqrt{a^2+\rho^2-2a\rho\cos\phi}} \rho^2 \, d\theta \, \sin\phi \, d\phi \, d\rho$$

$$\vec{r} = \begin{bmatrix} \rho\sin\phi\cos\theta \\ \rho\sin\phi\sin\theta \\ \rho\cos\phi \end{bmatrix}$$

$$I = 2\pi \int_0^R \rho^2 \lambda(\rho) \left[\int_{-1}^{1} \frac{dw}{\sqrt{a^2+\rho^2-2a\rho w}} \right] d\rho$$

$$= \frac{4\pi}{2a} \int_0^R \frac{\rho^2 \lambda(\rho)}{\rho} \left[-\sqrt{a^2+\rho^2-2a\rho w} \right]_{-1}^{1}$$

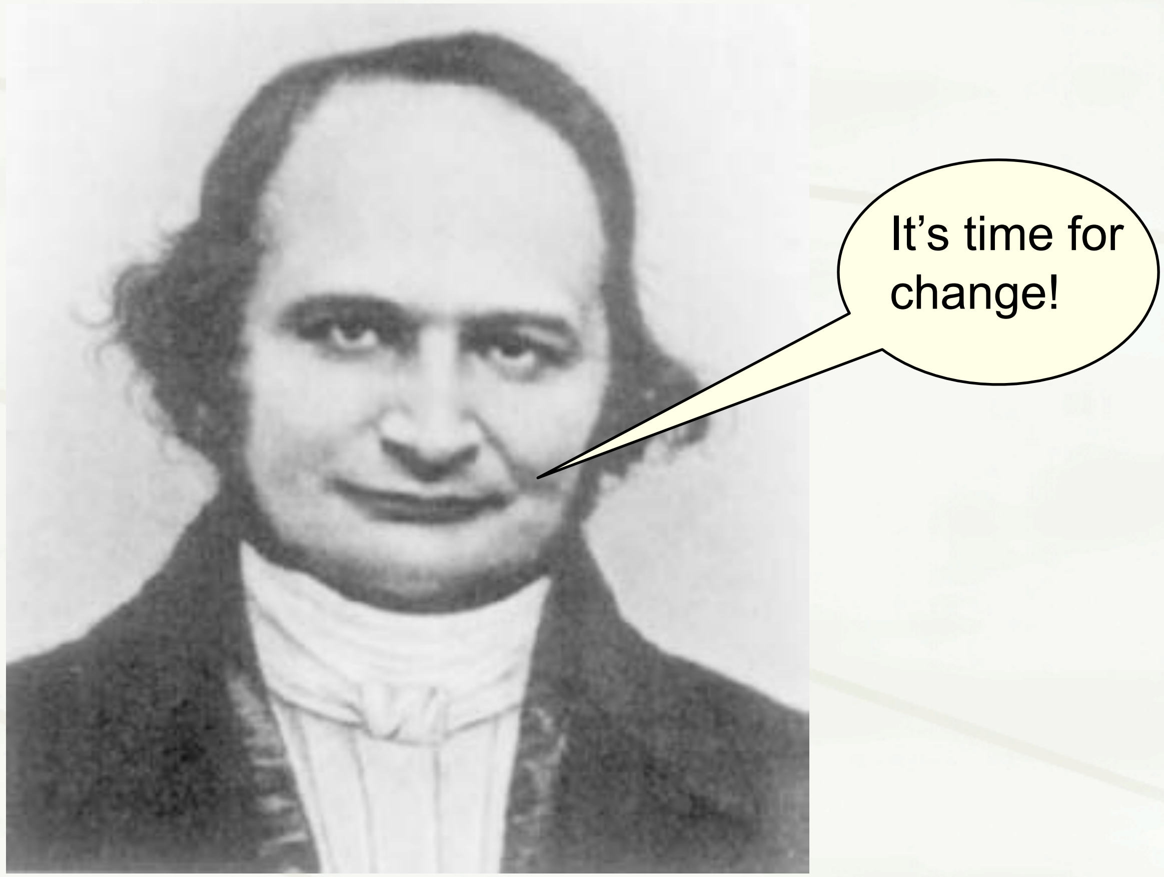
Another one of the great integrals

- Space travel would be a lot more complicated if gravitation didn't follow the inverse-square law to a high degree of accuracy.

The theme of today's lecture is…

Change!

The Great Variable Changer



Carl Gustav Jacob Jacobi
1804-1851

Sechsundzwanzigste Vorlesung.

Elliptische Coordinaten.

Die Hauptschwierigkeit bei der Integration gegebener Differentialgleichungen scheint in der Einführung der richtigen Variablen zu bestehen, zu

From Lecture 26 of the 8-th volume of Jacobi's collected lectures (p. 198): "The greatest difficulty in integrating differential equations seems to consist in introducing the right variables,…"

and not just Cartesian, polar, cylindrical, spherical.

Another interesting fact about Jacobi...

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Supported in part by a grant from The Clay Mathematics Institute.

Please email us with feedback.

Carl Gustav Jacob Jacobi

Biography

Ph.D. Humboldt-Universität zu Berlin 1825

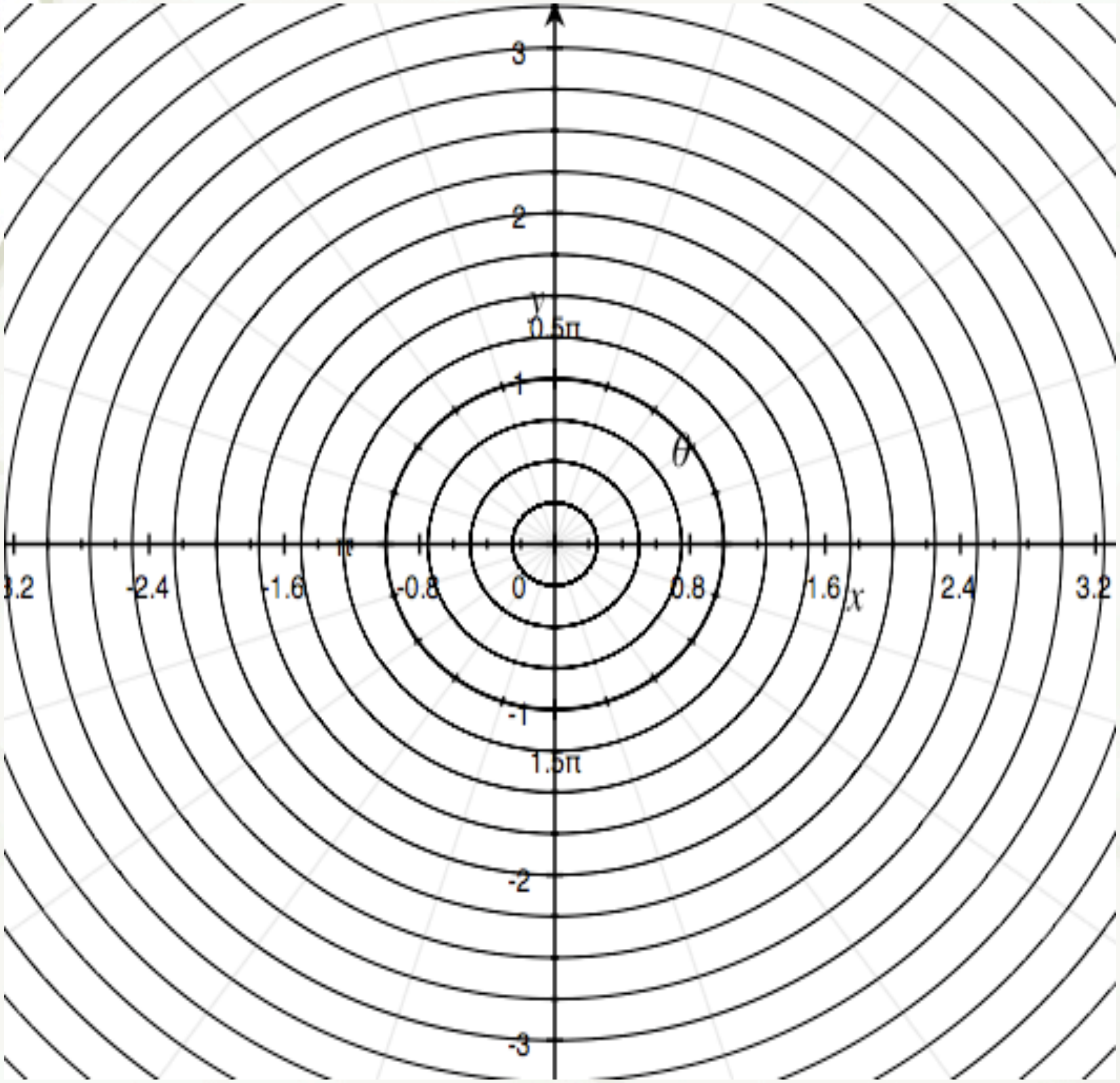
Dissertation: *Disquisitiones Analyticae de Fractionibus Simplicibus*

Advisor: Enno Dirksen

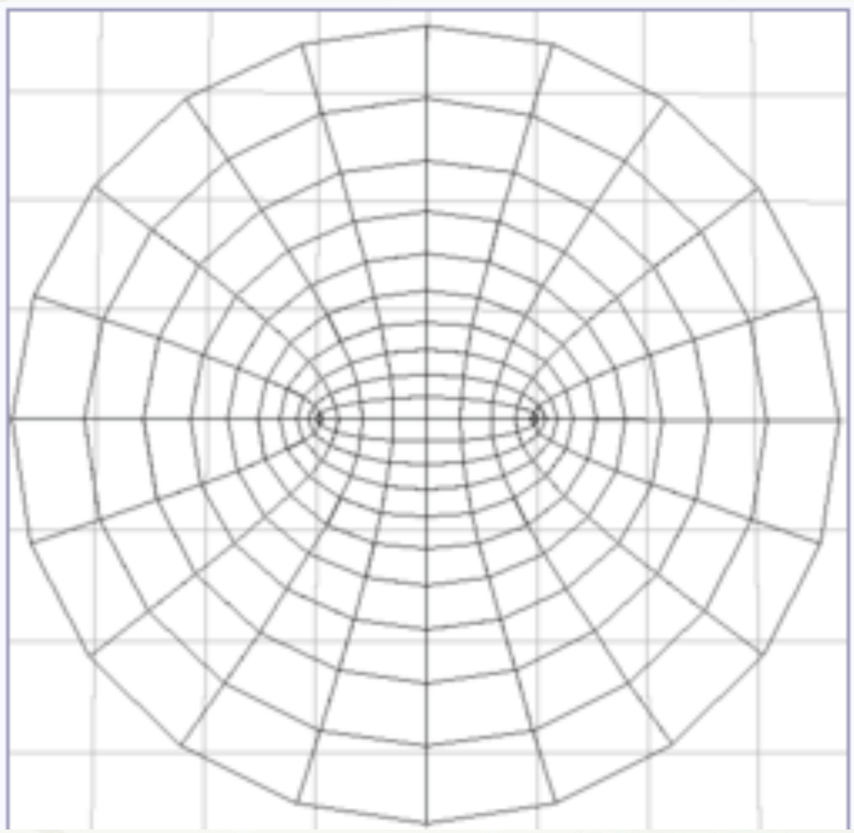
Student(s):
Click here to see the students listed in chronological order.

Name	School	Year	Descendants
Paul Gordan	Universität Breslau	1862	763
Oswald Hermes	Europa-Universität Viadrina Frankfurt an der Oder	1849	
Otto Hesse	Universität Königsberg	1840	5626
Friedrich Richelot	Universität Königsberg	1831	5525
Wilhelm Scheibner	Martin-Luther-Universität Halle-Wittenberg	1848	717

According to our current on-line database, Carl Jacobi has 5 students and 7420 descendants.



Polar coordinate lines



“Curvilinear” coordinates.

Example: elliptic

x = cosh u cos v

y = sinh u sin v

But what are u and v in terms of x and y?

Contest!

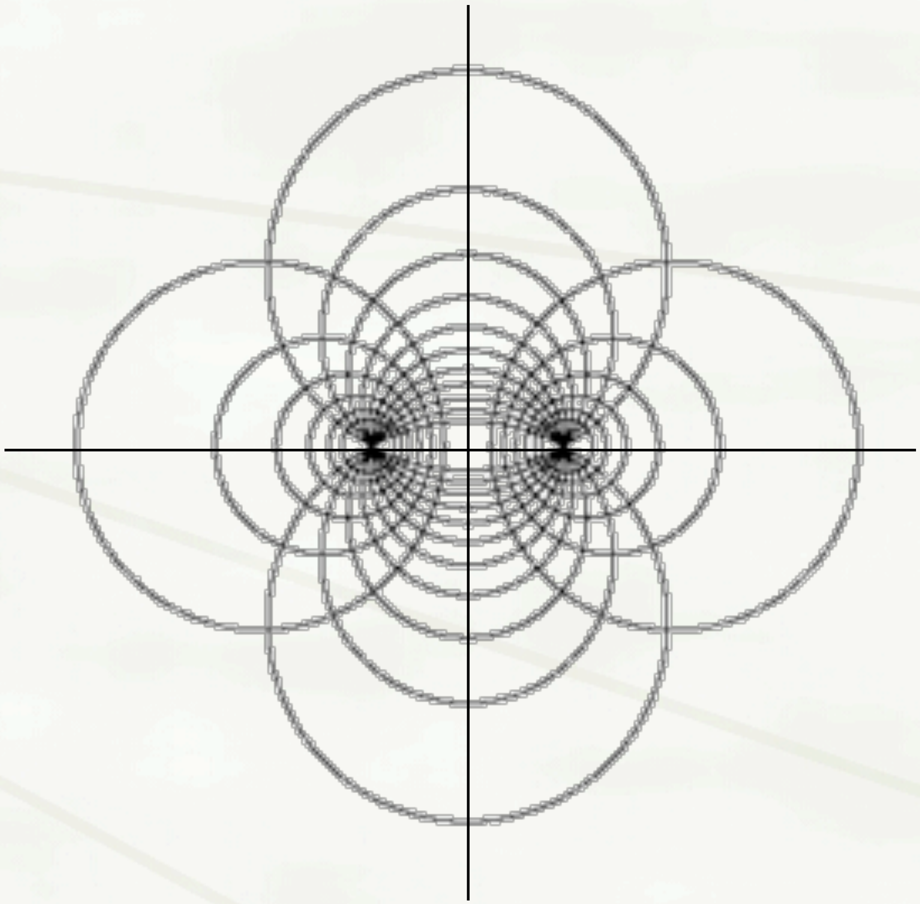
Another example: the bipolar coordinates.

$x = \sinh(t)/(\cosh(t)-\cos(s))$

$y = \sin(s)/(\cosh(t)-\cos(s))$

How do you like these coordinate curves?

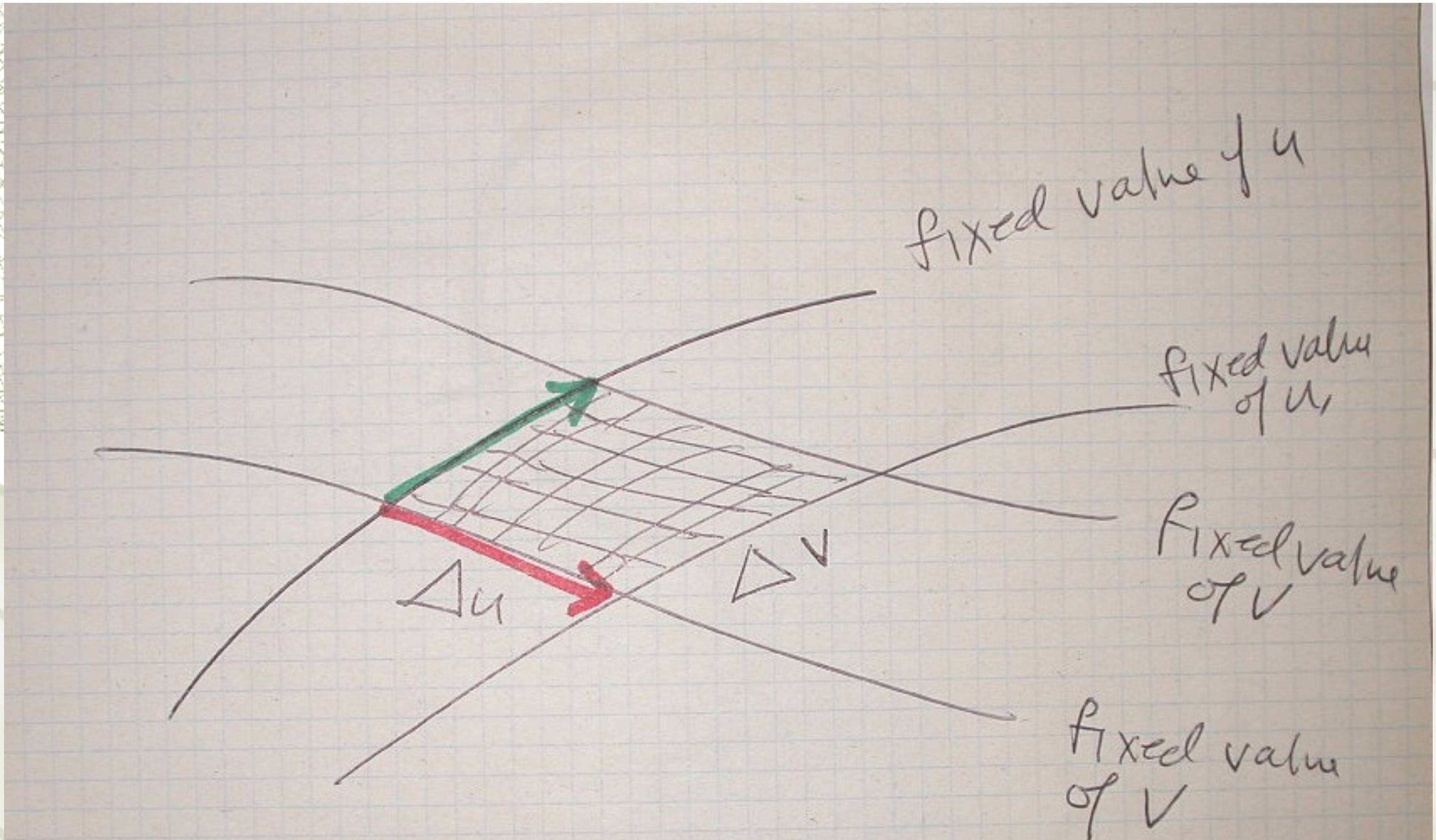
Bipolar coordinate lines:



(Adapted from graphic on Wikipedia.)

How would we integrate with curvilinear coordinates?

- Write $x = x(u,v)$, $y = y(u,v)$
  - Example: $x = r \cos \theta$, $y = r \sin \theta$.
- Make a "differential box" bounded by
  - $(x(u,v),y(u,v))$
  - $(x(u+\Delta u,v),y(u+\Delta u,v))$
  - $(x(u,v+\Delta v),y(u,v+\Delta v))$
  - $(x(u+\Delta u,v+\Delta v),y(u+\Delta u,v+\Delta v))$



→ is $\cong$ vector from $X(u,v)$ to $\vec{X}(u,v) + \frac{\partial \vec{X}}{\partial u}\Delta u$

→ is $\cong$ vector from $X(u,v)$ to $\vec{X}(u,v) + \frac{\partial \vec{X}}{\partial v}\Delta v$

$$\text{Area} \cong \left| \frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v} \right| \Delta u \Delta v$$

$$=: |J(u,v)| \Delta u \Delta v,$$

where

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

sometimes written: $\frac{\partial(x,y)}{\partial(u,v)}$

$$dx\,dy = \frac{\partial(x,y)}{\partial(u,v)}\, du\, dv$$

↑

Like chain rule in integration

Check on Jacobi with an example we know:

- Example: $x = r\cos\theta$, $y = r\sin\theta$.
  - $J(u,v) =$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

Therefore $dA = r \;\; dr\, d\theta$

Example

Integrate x+y over the region bounded by

$y = 3-x$

$y = 4-x$

$y = 2x$

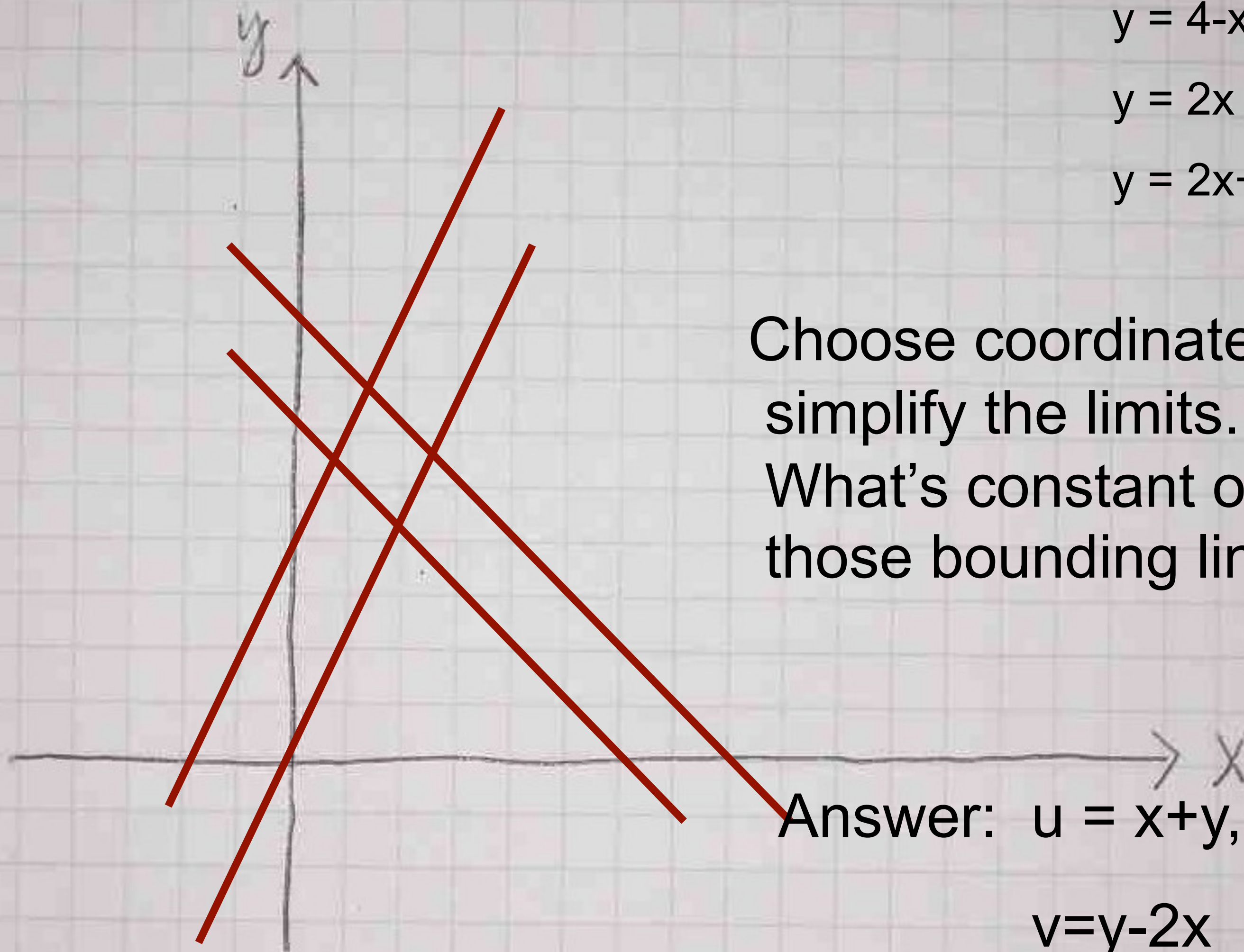
$y = 2x+2$

y = 3-x

y = 4-x

y = 2x

y = 2x+2



Choose coordinates to simplify the limits. What's constant on those bounding lines?

Answer: u = x+y,

v=y-2x

u goes from 3 to 4,

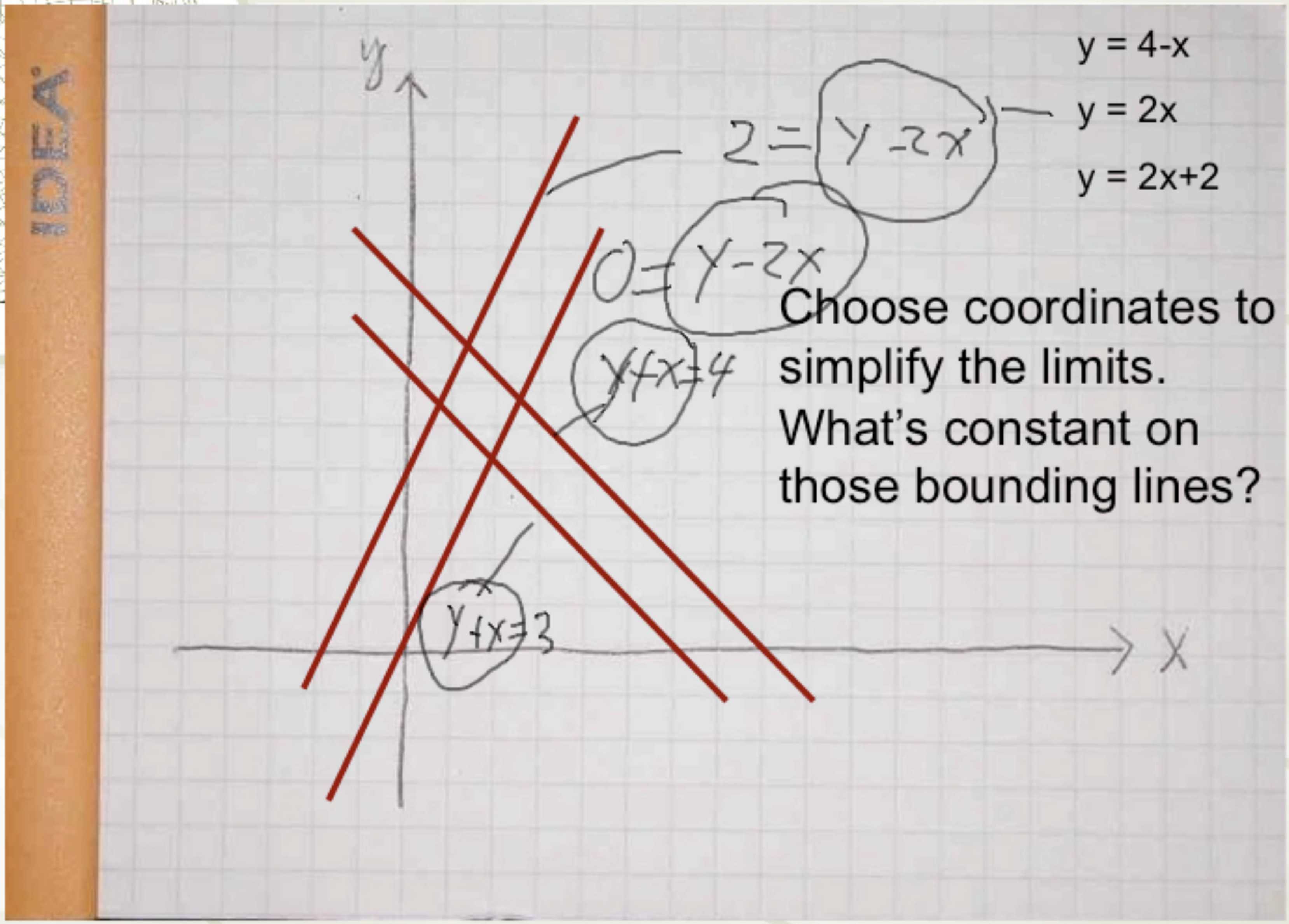
v goes from 0 to 2

y = 3-x

y = 4-x

y = 2x

y = 2x+2



Choose coordinates to simplify the limits. What's constant on those bounding lines?

$$\iint_R (x+y)\,dx\,dy \longrightarrow$$

$u = x + y$ (3 to 4)

$v = y - 2x$ (0 to 2)

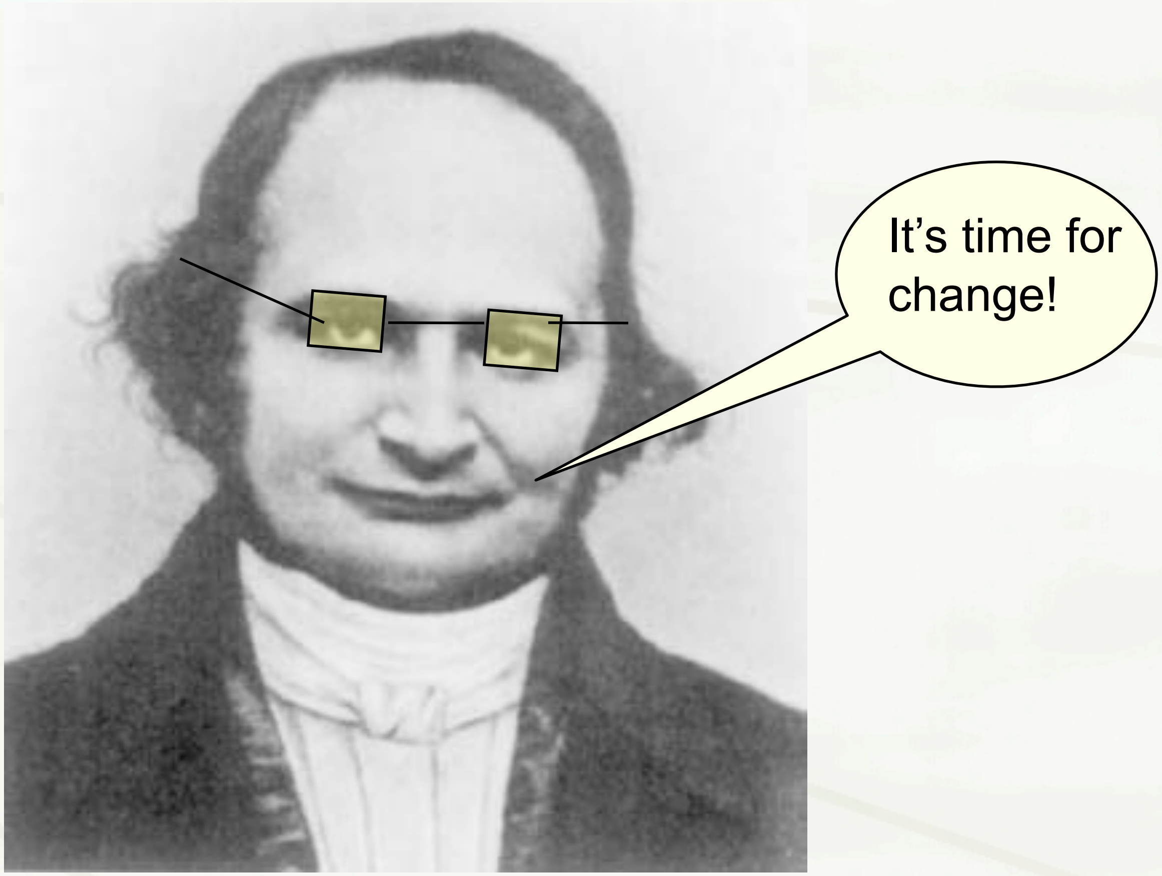
$u - v = 3x \qquad x = \left(\frac{u-v}{3}\right)$

$2u + v = 3y \qquad y = \left(\frac{2u+v}{3}\right)$

$$\int_0^2 \int_3^4 (u)\left(\frac{1}{3}\right) du\,dv$$

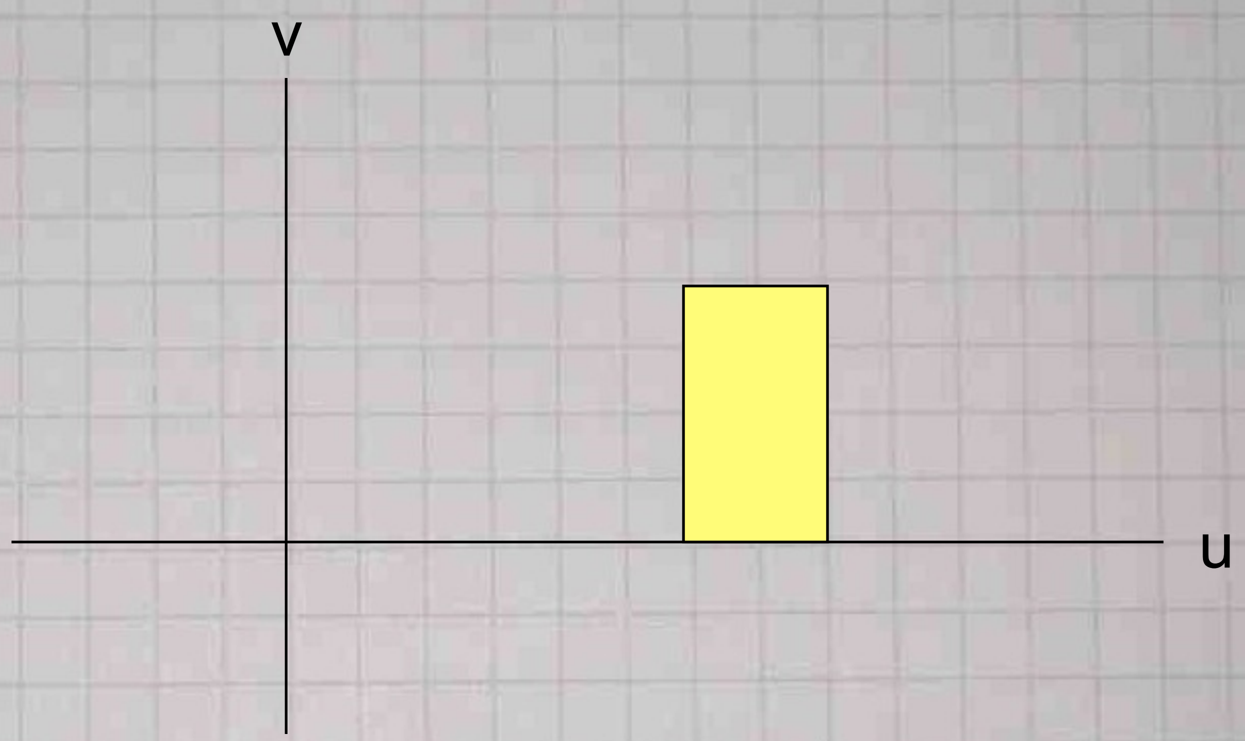
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}$$

$$= \frac{1}{3}\cdot\frac{1}{2}\left(4^2 - 3^2\right)(2-0) = \frac{14}{6} = \frac{7}{3}$$



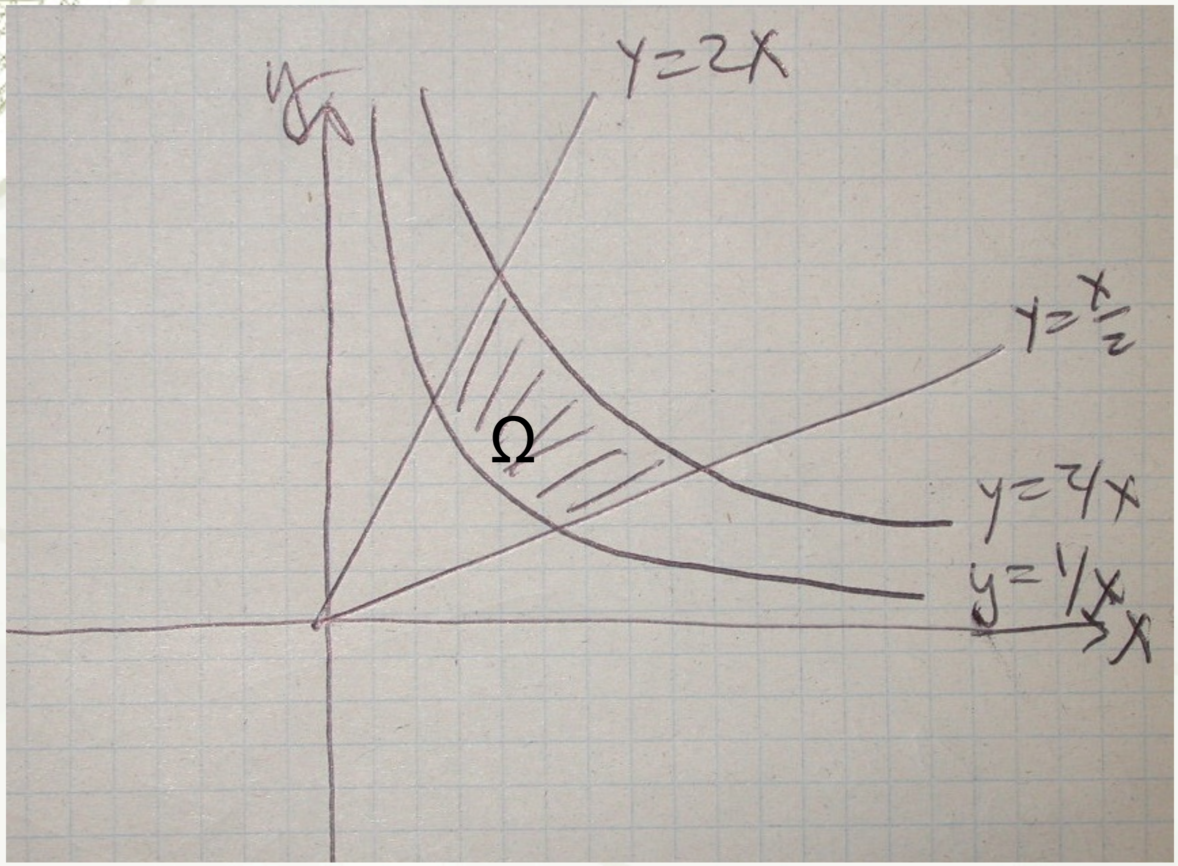
Special Jacobian glasses

To Jacobi, that region looks like this:



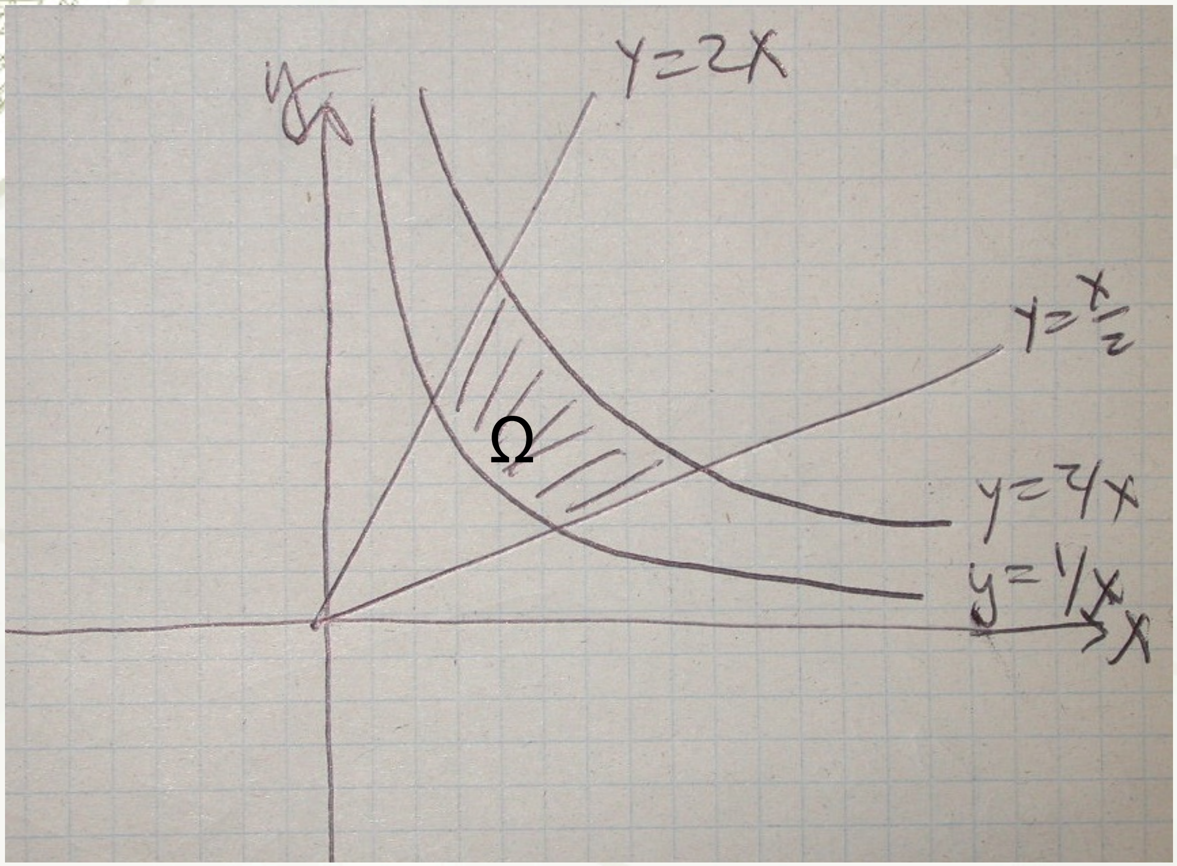
More examples, please

More examples, please



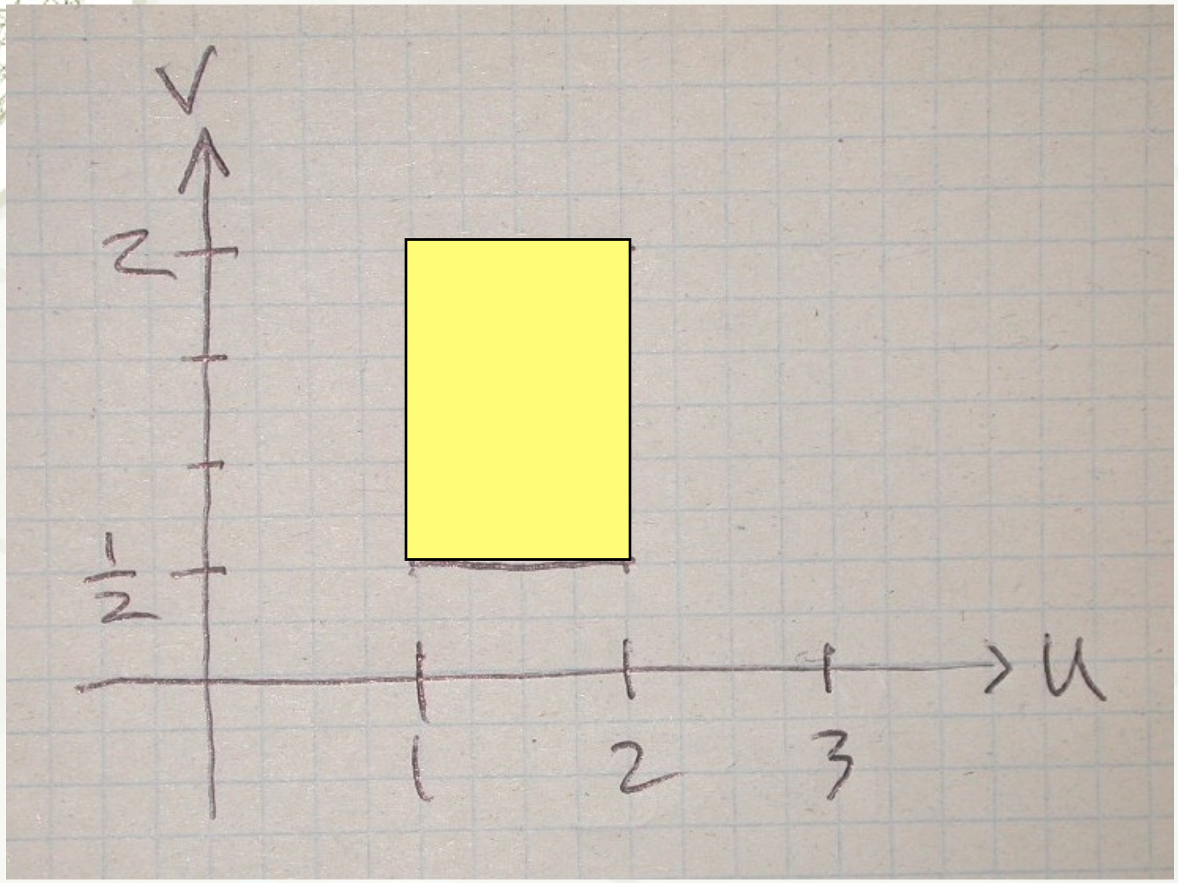
Calculate area and centroid.

You asked for it!



Good variables: $u = xy$, $v = y/x$

You asked for it!



Good variables: $u = xy$, $v = y/x$;

$x=(u/v)^{1/2}$, $y=(uv)^{1/2}$.

In the good variables:

$x=(u/v)^{1/2}$, $y=(uv)^{1/2}$, so

$J(u,v) = \ldots$

(Finish as exercise)