The return of the great variable changer

First… the puzzle of the day

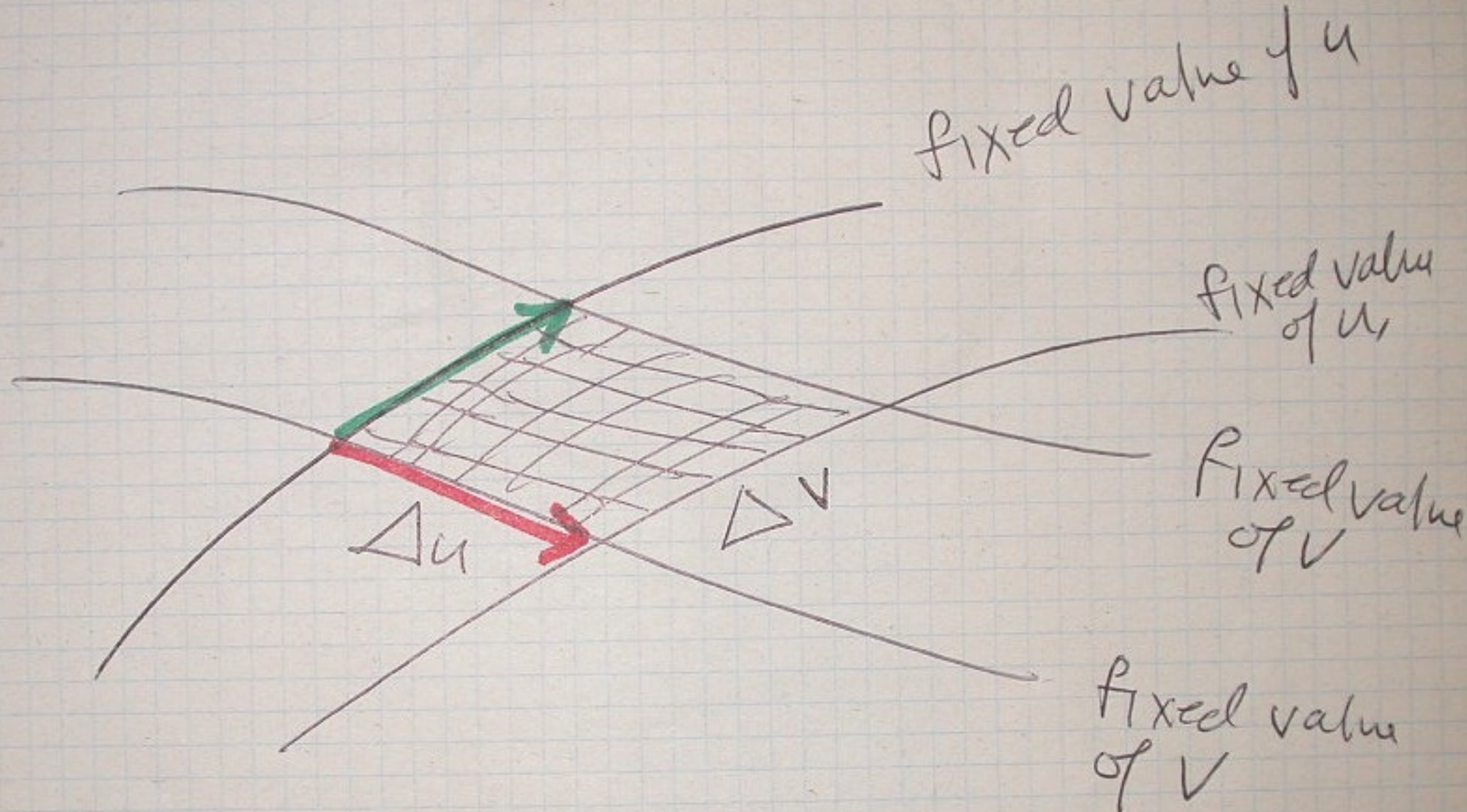
- $0 = 0 + 0 + 0 + \ldots\ldots$

  $= (1 – 1) + (1 – 1) + (1 – 1) + \ldots\ldots$

  $= 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \ldots\ldots$

  $= 1 + 0 + 0 + \ldots$

  $= 1$



→ is ≅ vector from $X(u,v)$ to $\vec{X}(u,v) + \frac{\partial \vec{X}}{\partial u}\Delta u$

→ is ≅ vector from $X(u,v)$ to $\vec{X}(u,v) + \frac{\partial \vec{X}}{\partial v}\Delta v$

$$\text{Area} \cong \left| \frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v} \right| \Delta u \Delta v$$

$$=: |J(u,v)| \Delta u \Delta v,$$

where

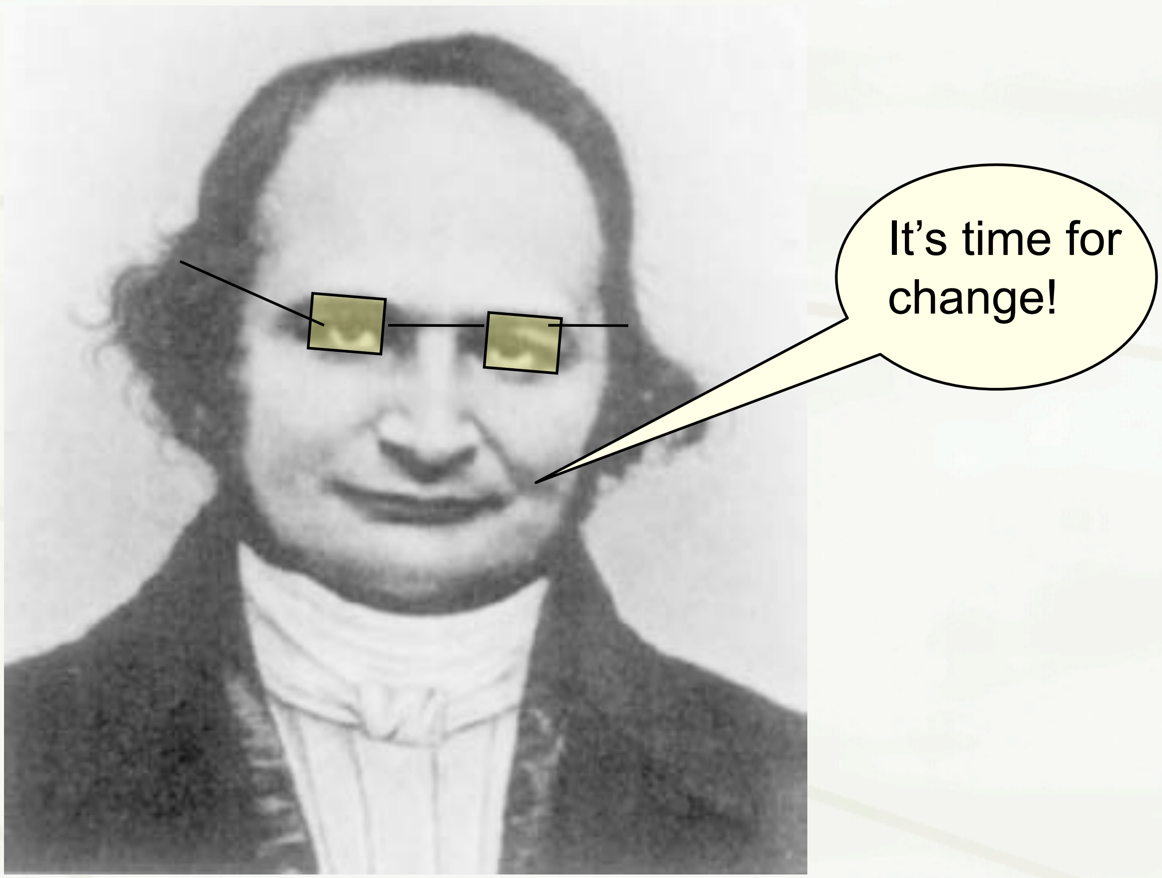
$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

sometimes written: $\frac{\partial(x,y)}{\partial(u,v)}$

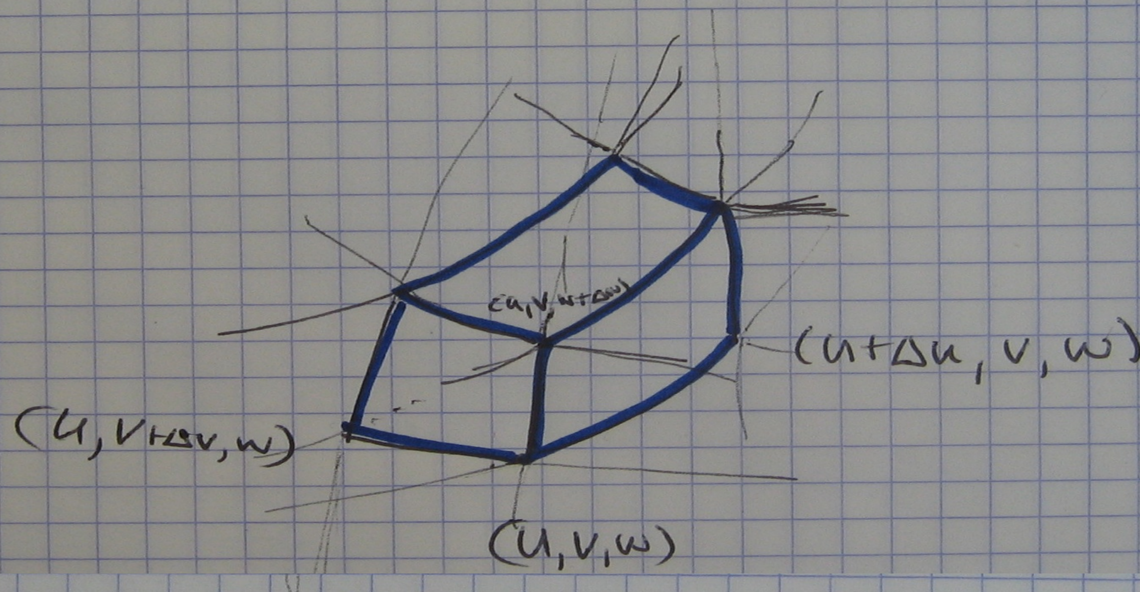
$$dx\,dy = \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$

↑

Like chain rule in integration



What can this man do in 3D?
….or 17D?



parallelopiped with edges $\frac{\partial \vec{r}}{\partial u}\Delta u$, $\frac{\partial \vec{r}}{\partial v}\Delta v$, and $\frac{\partial \vec{r}}{\partial w}\Delta w$.

Volume is $a\cdot(b\times c)$, same as $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$,

or, if $a=\frac{\partial \vec{r}}{\partial u}\Delta u$, etc.

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} =: \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

Jacobi in 3D

- $x=x(u,v,w)$, $y=y(u,v,w)$, $z=z(u,v,w)$,
  - Example:
    - $x = \rho \sin(\phi) \cos(\theta)$
    - $y = \rho \sin(\phi) \sin(\theta)$
    - $z = \rho \cos(\phi)$

- The volume of a little 3 D box is again a determinant, up to a sign.

Cartesian to spherical

$x \to \rho \sin\phi \cos\theta$

$y \to \rho \sin\phi \sin\theta$

$z \to \rho \cos\phi$

$$J = \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \rho\cos\phi\cos\theta & \rho\cos\phi\sin\theta & -\rho\sin\phi \\ -\rho\sin\phi\sin\theta & \rho\sin\phi\cos\theta & 0 \end{vmatrix}$$

← nice!

$$= \rho^2 \left\{ \sin^3\phi \sin^2\theta + \cos^2\phi \sin\phi \sin^2\theta + \sin^3\phi \cos^2\theta + \cos^2\phi \sin\phi \cos^2\theta \right\}$$

$$= \rho^2 \left\{ \sin^3\phi + \cos^2\phi \sin\phi \right\} = \boxed{\rho^2 \sin\phi}$$

Jacobi in 3D

- The Jacobian when you change variables from $\{x_1, \ldots x_n\}$ to $\{u_1, \ldots u_n\}$ is always the
  - absolute value of
  - the determinant of
  - the matrix

$$\{\partial x_i/\partial u_j\}$$

- Because this is the volume of an n-dimensional parallelopiped with sides $\partial \mathbf{r}/\partial u_j$.

This **r** is the position vector as a function of the u's!

Jacobi in 3D

- Example. Find the integral of $(x+y+z)^{1/2}$ on Ω bounded by
  - $0 \le x+y+z \le 16$,
  - $1 \le 2x+y \le 4$,
  - $1 \le y-3z \le 5$
- Hint: Do you really have to work out x,y,z in terms of u,v,w? How are $\partial(x,y,z)/\partial(u,v,w)$ and $\partial(u,v,w)/\partial(x,y,z)$ related?

Jacobi in 3D

- Example. Find the integral of $(x+y+z)^{1/2}$ on Ω bounded by
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- Hint: Do you really have to work out x,y,z in terms of u,v,w?

Nope. The two Jacobians are reciprocals. However, remember that one of them is a function of the old variables and the other is a function of the new variables.

$$u = x+y+z$$
$$v = 2x+y$$
$$w = y-3z$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -3 \end{vmatrix}$$

$$= 1\cdot(-3) - 2\cdot(-4) = 5$$

fn of x,y,z

Therefore $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{5}$, and

fn of u,v,w

$$\iiint_{\Omega} \sqrt{x+y+z}\, dx\,dy\,dz = \frac{1}{5}\int_1^5 \int_1^4 \int_0^{16} \sqrt{u}\, du\,dv\,dw$$

$$= \frac{1}{5}\left(\int_1^5 dw\right)\left(\int_1^4 dv\right)\left(\int_0^{16} \sqrt{u}\, du\right)$$

$$= \frac{1}{5} \cdot 4 \quad 3 \cdot \frac{2}{3}(16)^{\frac{3}{2}}$$

= 512/5

And now for something completely different

Chapter 18 – “line integrals”:

$$W = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

A different kind of integral in 3D: "line integral"

- Work done by a force on a moving object:

$$W = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

$$W = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

A worry: what if two people parametrize the curve differently. Is the result the same?

A different kind of integral in 3D: "line integral"

- Work done by a force on a moving object:

$$W = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \;\frac{dt}{ds}\, ds$$

$$= \int_c^d F(\vec{\tilde{r}}(s)) \cdot \frac{d\vec{\tilde{r}}(s)}{ds}\, ds$$

A worry: what if two people parametrize the curve differently. Is the result the same?

Examples

- In all cases, let the curve be the unit circle, traversed counterclockwise.

1. $\mathbf{F}(x) = x\,\mathbf{i} + y\,\mathbf{j}$
2. $\mathbf{F}(x) = y\,\mathbf{i} + x\,\mathbf{j}$
3. $\mathbf{F}(x) = y\,\mathbf{i} - x\,\mathbf{j}$

Examples

- In all cases, let the curve be the unit circle, traversed counterclockwise.

1. **F**(x) = x **i** + y **j**

   - If x = cos t, y = sin t, then d**r** = (-sin t **i** + cos t **j**) dt, then **F** · d**r** = 0 at each point.

For a practical consequence of this fact, look at http://www.youtube.com/watch?v=osjec5k1oLI

Examples

- In all cases, let the curve be the unit circle, traversed counterclockwise.

1. $\mathbf{F}(x) = x\,\mathbf{i} + y\,\mathbf{j}$

2. $\mathbf{F}(x) = y\,\mathbf{i} + x\,\mathbf{j}$
   - If $x = \cos t$, $y = \sin t$, then $d\mathbf{r} = (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})\,dt$, then $\mathbf{F} \cdot d\mathbf{r} = (-\sin^2 t + \cos^2 t)\,dt = \cos(2t)\,dt$.
   - The integral of this for t from 0 to $\pi/2$ is $\pi/4$. If we integrate around the whole circle, so the beginning and end points are the same, we get 0.

Examples

- In all cases, let the curve be the unit circle, traversed counterclockwise.

1. **F**(x) = x **i** + y **j**

2. **F**(x) = y **i** + x **j**

3. **F**(x) = y **i** - x **j**. This time $\mathbf{F} \cdot d\mathbf{r} = (-\sin^2 t - \cos^2 t)\, dt = -dt$.
   The integral around the whole circle is -2π, even though the beginning and end points are the same.

   Under what conditions is it true that an integral from **a** to **b**=**a** gives us 0 - as in one dimension?

*Under what conditions is it true that an integral from **a** to **b=a** gives us 0 - as in one dimension?*

Related questions:

1. Is there such a thing as an antiderivative?

2. Does the value of the integral depend on the path you take?

A mystical picture to contemplate.